Lecture 4

Chaos

Chaos and the butterfly effect.

Credit: University of Michigan

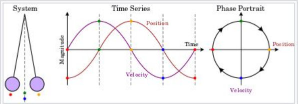


Fig. 1a. (from Wikipedia article Phase Space) A phase space of position and velocity for a simple pendulum.

Phase-space (position-velocity) diagram of a pendulum. Credit: Wikipedia

- Non-linear pendulum without dissipation or driving force

$$\ddot{\theta} = -\frac{g}{l}\sin\theta$$

non-linear: do **not** make the small angle approximation, $\sin\theta \approx \theta$

- No dissipation, no driving : energy conservation

and hence amplitude is constant as well

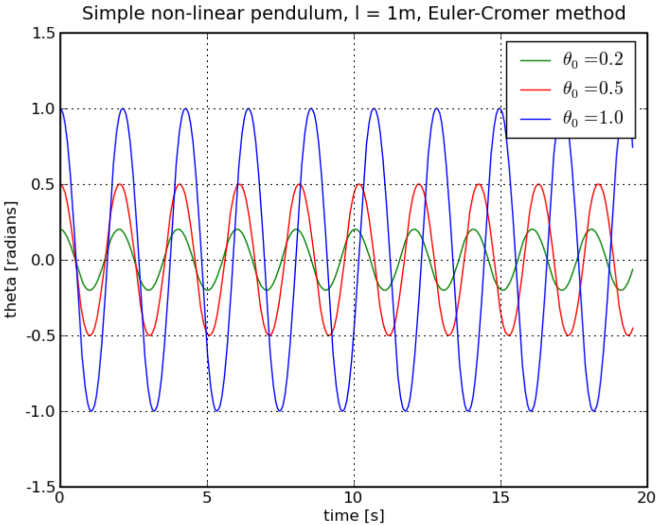
- Motion is periodic, but not simple harmonic

meaning: is **not** described by linear combination of $\sin(\Omega t)$ and $\cos(\Omega t)$

- Frequency depends on amplitude

unlike the case of simple harmonic oscillation, for which frequency $\Omega = \sqrt{\frac{g}{l}}$,is independent of θ_0 and ω_0

Simple non-linear pendulum



Driven, non-linear pendulum, with dissipation

- Add driving force and dissipation

$$\ddot{\theta} = -\frac{g}{l}\sin\theta + F_D \sin(\Omega_D t) - q\dot{\theta}$$

- Numerical solution: Euler-Cromer method
- θ coordinate has 'periodic boundary' conditions

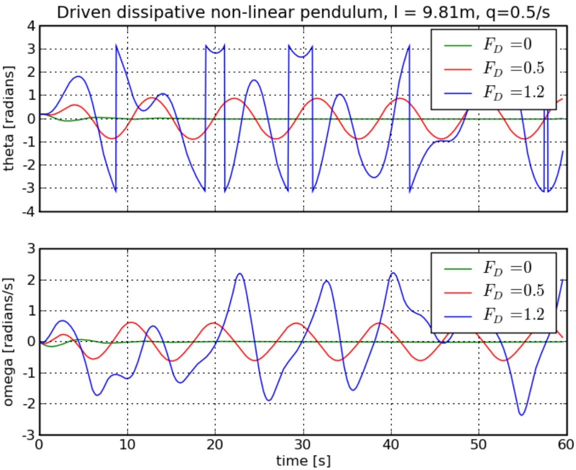
  meaning $\theta = \pi$ is same position of pendulum as $\theta = -\pi$, for example

  may lead to 'jumps' in a plot of $\theta(t)$ vs time t
- When $F_D = 0$ but $q > 0$: amplitude decreases with time
- When $F_D > 0$: Different regimes:
  - pendulum in resonance with driving force

    frequency is Ω_D, amplitude may increase
  - $\theta(t)$ plot may appear chaotic

    subject of this lecture

Driven, non-linear pendulum, with dissipation



damped oscillation, driven: $\Omega = \Omega_d$, no apparent periodicity

$\theta \to -\pi \to \pi$: pendulum *rotates* rather than oscillates

Driven, non-linear pendulum, with dissipation

- $\theta(t)$ for $F_D = 0.5$ is periodic but $F_D = 1.2$ is chaotic

  no apparent periodicity, even at much later times: we call this chaos
- In what sense is this 'chaotic'?
  - $\theta(t)$ appears to be 'unpredictable' no obvious pattern emerges
  - Yet solution is determined uniquely by the DE and its initial condition
- Example of **deterministic chaos** seems like a contradiction in terms

  Small differences in initial conditions get amplified

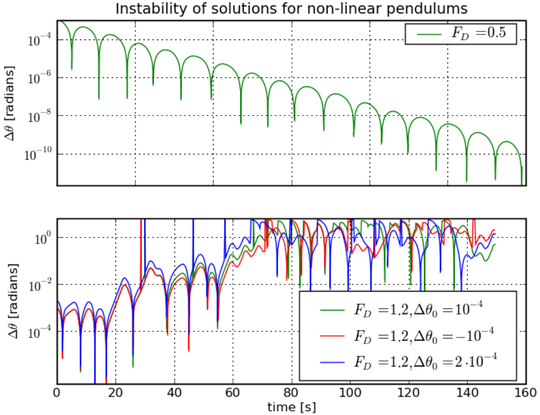
  $|\theta_1(t=0) - \theta_2(t=0)| < \epsilon \to |\theta_1(t) - \theta_2(t)| \gg \epsilon$

  Generic outcome of non-linear DEs.

  Arguably this also implies we cannot really obtain the numerical answer, since small errors build-up catastrophically
- More careful analysis subject of this lecture.

Chaos: dependence on initial conditions

- Compare evolution for small change in initial value $\theta_0 = 0.2 \pm \Delta\theta_0$



$F_D = 0.5$: small differences in ICs stay small in fact become smaller

$F_D = 1.2$: small differences in ICs amplify rapidly saturate at $\Delta\theta = \pi$ - no correlation between θ_1 and θ_2

Chaos: dependence on initial conditions

Lyapunov exponents

- Vary start condition (initial displacement $\theta_0 = 0.2$)
- Compute evolution of two identical pendulums, differing by $\Delta\theta_0 = \mathcal{O}(0.0001)$
- Plot difference $\Delta\theta = \left|\theta^{(1)} - \theta^{(2)}\right|$ as function of time.
- Findings:
  - For $F_D = 0.5$ dampening dominates and $|\Delta\theta|$ decreases.
  - For $F_D = 1.2$ $|\Delta\theta|$ increases (up to max$=\pi$)

  In both cases for t small: $|\Delta\theta| \sim e^{\lambda t}$ $\lambda < 0$ not chaotic, $\lambda > 0$: chaotic
- λ is called Lyapunov exponent

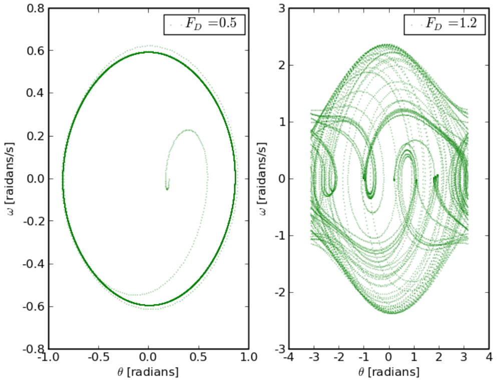
  Simple test: $\lambda > 0 \implies$ chaotic, $\lambda < 0 \implies$ not chaotic
- Definition of deterministic chaos:
  System shows deterministic chaos, if its evolution depends sensitively on the initial conditions

Mathematically implying it has $\lambda > 0$. For pendulum: if driving force dominates over damping

Visualising chaos: Phase space

phase space is position-velocity space. For pendulum: $(\theta, \omega = \dot{\theta})$ space

- Plot trajectories in θ-ω space.



Non-chaotic case (left) looks strikingly different from chaotic case (right). Yet there is still some structure in the right panel. Question: how would a periodic pendulum look like in this plot?

Note: curve is composed of many dots, since it is integrated numerically

Visualising chaos: Phase space

- Small driving force left panel
  - Transient time at beginning: eigen-frequency decays

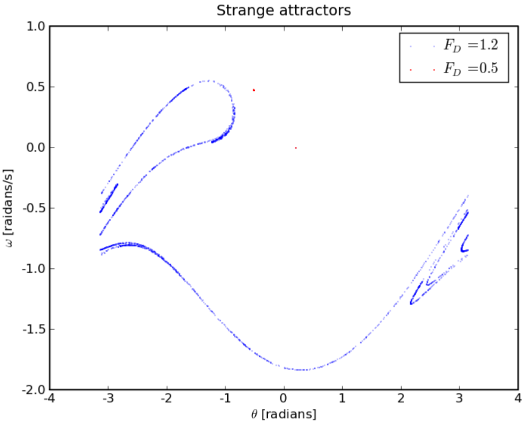
    different frequency corresponds to different ω at a given θ
  - pendulum quickly settles into a regular orbit;
  - shape of $\omega(\theta)$ curve is independent of initial conditions (in agreement with $\lambda < 0$)
- Large driving force: right panel
  - Expectation of no structure in this panel is **not true!**

    Notice that there is no maximum θ: pendulum goes all the way around
  - Surprise: recognizable orbits, even though chaotic

    but a given orbit is traversed only a few times
  - Examine phase space by plotting its Poincaré section
    Plot ω vs θ but only when $\Omega t = 2n\pi$, with $n \in N$

    meaning: plot position in phase space *when the driving term is zero but increasing*

Chaos: Poincaré section



value of $\omega(\theta)$ when driving force is zero: Poincaré section

non-chaotic: just two points (shown as red dots)

chaotic: a curve called **strange attractor**

Chaos: A section on Poincaré

Henri Poincaré.

Credit: Wikipedia

French mathematician, 1854-1912. Worked on many things, including the three-body problem.

Chaos: Strange attractors

- Poincaré section is very different for chaotic versus non-chaotic motion
  - non-chaotic: just a few points
    original motion at eigen frequency, the driven motion with $\Omega = \Omega_d$
  - chaotic: a fuzzy surface
    - 'Fuzziness' is **not** due to numerics property of the system
    - Shape of surface is largely independent of initial conditions

      important: implies that the Poincaré section is a good way to examine deterministic chaos
    - A fractal structure (fractals discussed again in lecture 7)

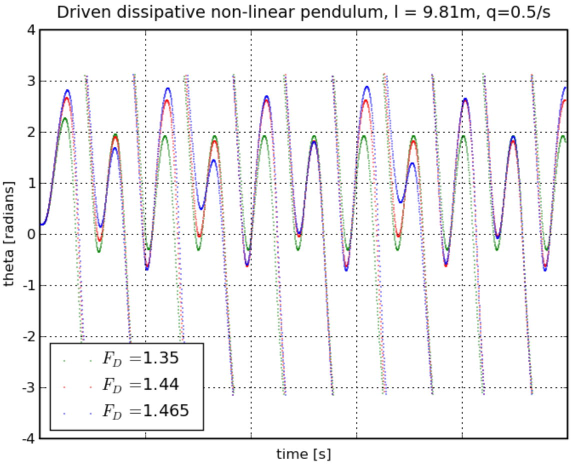
Transition to chaos: Period doubling

- What happens to solution when $F_D = 0.5 \to 1.2$ non-chaotic $\to$ chaotic?
- Answer: Not only one, but a chain of transitions: **Hard to study**
- Therefore: look for $F_D \in [1.3, 1.48]$ (fix $\Omega_D = 3\pi$).

  nature of the transition is clearer for this choice of F_D
- In this region, the period starts doubling!
  $\Longrightarrow$ periodic motion with frequencies $\Omega_D/2$, $\Omega_D/4$ etc..
- Typically the **opposite** happens in a harmonic oscillator

  when harmonics appear - oscillations with period $P/2$, $P/3$, etc. - e.g. in string instruments
- In the chaotic pendulum, periodicities with period $2P$, $4P$ etc appear - **sub- harmonics** P is the period of the driving force

Transition to chaos: Period doubling pendulum

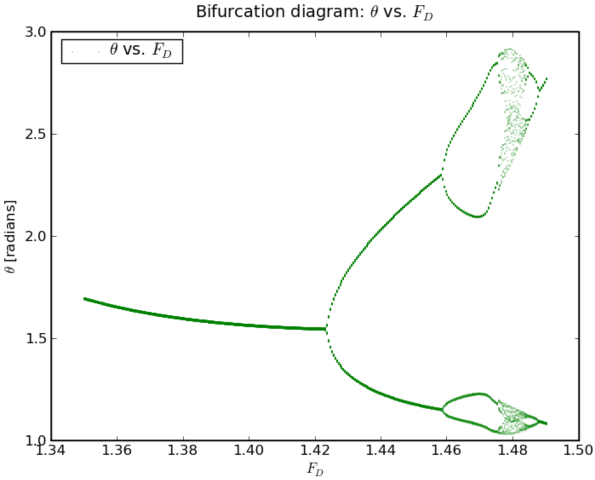


Amplitude of the second maximum is smaller - true (almost periodic) cycle has a period **twice** that of the driving force (for $F_D = 1.44$) - and $4P$ (for $F_D = 1.465$)

Transition to chaos: Bifurcations

way to visualise how period doubling leads to chaos

- Plot θ when $\Omega_D t = 2\pi\, n$ with $n \in N$ *i.e.* at given phase in driving cycle



Interpreting the bifurcation diagram

- At low F_D left side of diagram

  At a given phase of the driving force, pendulum is at a single value of θ
- At $F_D \approx 1.43$: First period doubling

  two possible values for θ at a given phase of driving, small or large amplitude oscillation
- At $F_D \approx 1.46$: Second period doubling

  four possible values for θ at a given phase of driving - 4 different possible amplitudes
- At larger F_D - more and more period doublings appear
- Introduce $F_n = F_D$ for nth period doubling and

$$\delta_n = \frac{F_n - F_{n-1}}{F_{n+1} - F_n} .$$

  In limit of $n \to \infty$, $\delta_n \to \delta_\infty \approx 4.669$
- Universal feature: δ_∞ seemingly the same for all systems

  for which where period doubling leads to chaos, see Feigenbaum's original 1987 paper

Summary

- Concept of deterministic chaos

  for example, non-linear, damped, driven pendulum

- Ways to quantify/describe chaos:
  - Lyapunov exponents exponential divergence of solutions with nearly ICs
  - Phase space diagrams and strange attractors

    way to visualise behaviour

  - Period doubling leading to chaos $\Longrightarrow$ bifurcations

    way to visualize period doubling

- Clearly very hard to do something like this analytically!
- Unexpected behaviour of 'simple' non-linear equations

  hot research topic when computers became readily available