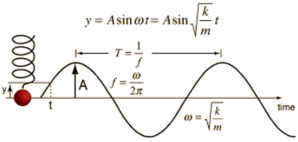
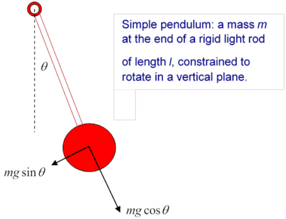
Lecture 3:
Harmonic Motion



Bob on a spring Credit: Oregon State University



A simple pendulum. Credit: Viriginia University

Mathematical model & analytic solution

- Force is proportional to displacement:

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -k\,x$$

k is a constant, m is mass of object

$k > 0$: minus sign results in a restoring force (oscillations)

Numerous physics examples, e.g. pendulum when angle is small, Hook's law for bob on a spring

2^{nd} order DE: need to specify $x(t=0)=x_0$, $\dot{x}(t=0)=\dot{x}_0$

- Analytical solution

$$\begin{aligned} x(t) &= A\cos(\Omega t) + B\sin(\Omega t)\,; \quad \Omega^2 = \frac{k}{m} \\ x_0 &= A\,; \quad \dot{x}_0 = \Omega\,B \end{aligned}$$

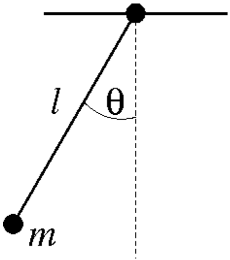
Initial conditions determine A and B

Example of harmonic motion: pendulum

- Pendulum bob of mass m attached to a (rigid & massless) rope of length l, θ is deflection angle from vertical
- Consider components of gravitational force, $m\mathbf{g}$ along and perpendicular to rope. Component along rope balanced by rope's tension. Component perpendicular is

$$F_\theta = -mg\sin\theta \approx -mg\theta$$

  in the small angle approximation.



- Apply Newton's law:

$$m\ddot{r} = ml\ddot{\theta} = -mg\theta\,; \quad \ddot{\theta} = -\frac{g}{l}\theta$$

$$m\ddot{x} = -k\,x\,; \quad x = \theta \;\&\; \frac{k}{m} = \frac{g}{l} = \Omega^2$$

Example of harmonic motion pendulum (cont'd)

- Analytical solution small angles: $\theta(t) = A\cos(\Omega t) + B\sin(\Omega t)$
- Angular eigen-frequency: $\Omega = \sqrt{\frac{g}{l}}$
- Choose initial conditions: $t = 0$ corresponds to θ is maximum
  - Maximal amplitude: $\theta = \theta_0$ when $t = 0 \rightarrow A = \theta_0$.
  - Angular velocity $\omega \equiv \dot{\theta} = 0$ when $t = 0 \rightarrow B = 0$
- Energy E of pendulum is conserved: meaning it is constant

$$
\begin{aligned}
E &= \frac{1}{2}ml^2\omega^2 + mgl(1-\cos\theta) \approx \frac{1}{2}ml^2\omega^2 + \frac{1}{2}mgl\theta^2 \\
\dot{E} &= ml\omega(l\dot{\omega} + g\theta) = 0\,; \quad \text{since} \;\; \dot{\omega} = \ddot{\theta} = -\frac{g}{l}\theta
\end{aligned}
$$

$1 - \cos(\theta) \approx \theta^2/2$ in the small angle approximation

Numerical solution: Euler's method

- As in lecture 2: replace 2$^{\mathrm{nd}}$ order DE by two 1$^{\mathrm{st}}$ order DEs and solve using Euler's method

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\frac{g}{l}\theta \rightarrow \frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega\,; \quad \frac{\mathrm{d}\omega}{\mathrm{d}t} = -\frac{g}{l}\theta$$

- Discretise: $dt \rightarrow \Delta t$

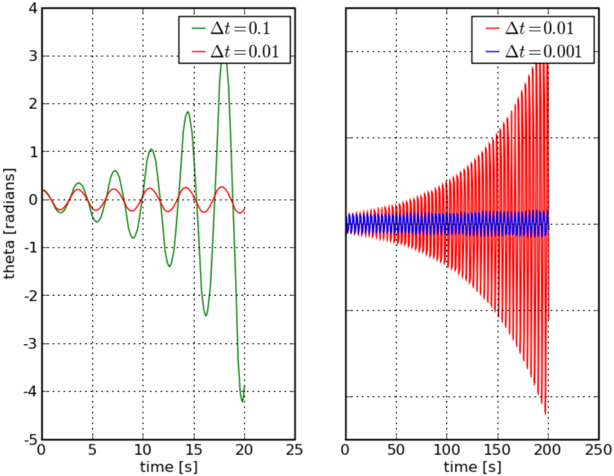
$$\begin{aligned}
\theta^{n+1} &= \theta^n + \omega^n \Delta t \\
\omega^{n+1} &= \omega^n - \frac{g}{l}\theta^n \Delta t \\
t^{n+1} &= t^n + \Delta t
\end{aligned}$$

- Choose time-step to be small compared to speriod: $\Delta t \ll 2\pi/\Omega$

Numerical solution: Euler's method (cont'd)

- Problem: Amplitude increases with time even for small Δt

(just need to run long enough . . .)



Euler's method: why does it fail?

- Increasing amplitude implies energy of numerical solution increases whereas energy should be constant!
- Evaluate numerical energy: recall: $E = ml^2\omega^2/2 + mgl\theta^2/2$

$$
\begin{aligned}
E^{n+1} &= \frac{ml^2}{2}\left[(\omega^{n+1})^2 + \frac{g}{l}(\theta^{n+1})^2\right] \\
&= \frac{ml^2}{2}\left[\left(\omega^n - \frac{g}{l}\theta^n\Delta t\right)^2 + \frac{g}{l}\left(\theta^n + \omega^n\Delta t\right)^2\right] \\
&= E^n + \frac{mgl}{2}\left(\frac{g}{l}(\theta^n)^2 + (\omega^n)^2\right)\Delta t^2 \\
&> E^n
\end{aligned}
$$

for any choice of time-step

- Numerical scheme does not conserve energy!

Euler's method: why does it fail (con't)

- Euler method not good for harmonic motion.
- Okay, fine, but why was it good before? Was energy conserved applying Euler's method to ballistic motion?

  Euler's method violates energy conservation of cannon ball - as does Runge-Kutta method

  Remember the trajectory of the cannon ball: For larger step-size higher peak in trajectory than for smaller step-size (with roughly the same range)
- in practise: only calculate parabolic trajectory (cannon ball) compared to many oscillations (harmonic motion) Euler's method OK for trajectories - but not for harmonic motion there may be exceptions, for example planetary orbits - need to compute many cycles
- There is no single method that is perfect for all problems.

Improving the Euler method: Euler-Cromer

- Obvious solution: use Runge-Kutta instead

  energy conservations is better for same Δt - but still not perfect!

- However, consider following small change to Euler's method: Instead of Euler's method

$$\omega^{n+1} = \omega^n - \frac{g}{l}\theta^n \Delta t \quad \text{and} \quad \theta^{n+1} = \theta^n + \omega^n \Delta t$$

use small change

$$\omega^{n+1} = \omega^n - \frac{g}{l}\theta^n \Delta t \quad \text{and} \quad \theta^{n+1} = \theta^n + \omega^{n+1} \Delta t$$

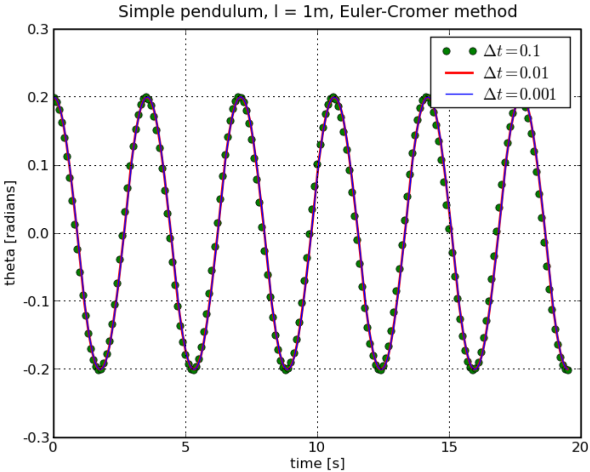
that is: use *new* value of ω to update θ

- Exercise: compute $E^{n+1} - E^n$

  $E^{n+1} - E^n = ((\omega^n)^2 - (\frac{g}{l}\theta^n)^2)\Delta t^2 - 2\frac{g}{l}\theta^n\omega^n\Delta t^3 + (\frac{g}{l}\theta^n)^2\Delta t^4$

Results with Euler-Cromer

- Amplitude does not increase rapidly, even if Δt is no very small!



Damping: mathematical model

- Damping slows down the pendulum bob:

  e.g. due to friction, or air resistance. Friction may depend on other powers of velocity too

$$\ddot{\theta} = -\Omega^2\theta \rightarrow \ddot{\theta} = -\Omega^2\theta - q\dot{\theta}$$

  q is taken to be a positive constant

- Form of analytical solution depend on value of q verify solutions
  1. Under-damped regime: amplitude decays exponentially $q < 2\Omega$

$$\theta(t) = \theta_0 \exp\left(-\frac{qt}{2}\right) \sin\left(\sqrt{\Omega^2 - q^2/4} \cdot t + \phi\right)$$

  2. Over-damped regime: no oscillations $q > 2\Omega$

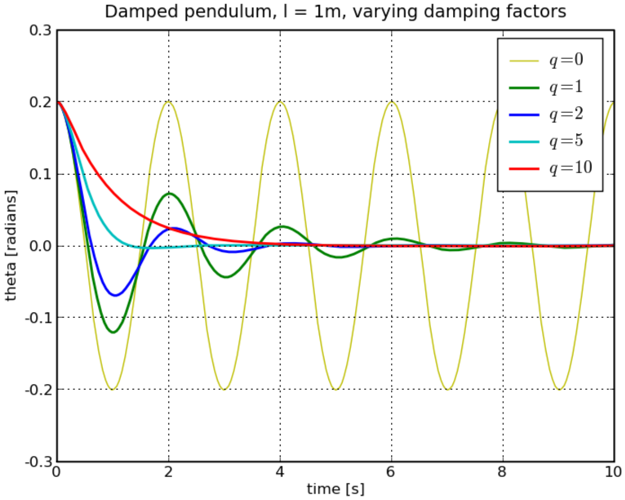
$$\theta(t) = \theta_0 \exp\left[-\left(\frac{q}{2} + \sqrt{q^2/4 - \Omega^2}\right) \cdot t\right]$$

  3. Critically damped regime: Pendulum "crawls" to 0 $q = 2\Omega$

$$\theta(t) = (\theta_0 + Ct) \exp\left(-\frac{qt}{2}\right).$$

Damping: numerical solution

- Amplitude decreases with time



Driven oscillation: mathematical model

- Add a time-varying force

$$\begin{aligned}\ddot{\theta} &= -\Omega^2\theta - q\dot{\theta} \quad \text{without driving force}\\ \ddot{\theta} &= -\Omega^2\theta - q\dot{\theta} + F_d \sin(\Omega_D t) \quad \text{driving force}\end{aligned}$$

strictly speaking, F_D is an acceleration, not a force – we will still call it force

driving force has amplitude $F_D > 0$ and varies sinusoidally with constant frequency Ω_D

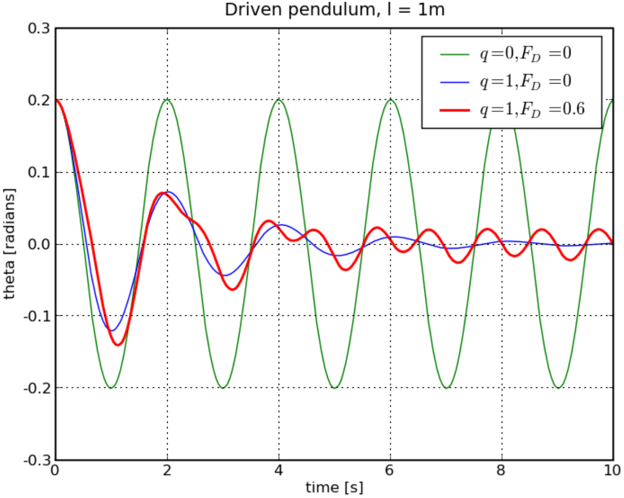
- Driving increases energy of the system.
  After initial transient:
  - frequency changes $\Omega \to \Omega_d$
  - amplitude changes

Analytical solution after transient

$$\begin{aligned}\theta(t) &= \theta_{\max} \sin(\Omega_D t + \phi)\\ \theta_{\max} &= \frac{F_D}{\sqrt{(\Omega^2 - \Omega_D^2)^2 + (q\Omega_D)^2}}\end{aligned}$$

Driven oscillation: numerical solution

- Frequency changes $\Omega \to \Omega_d$

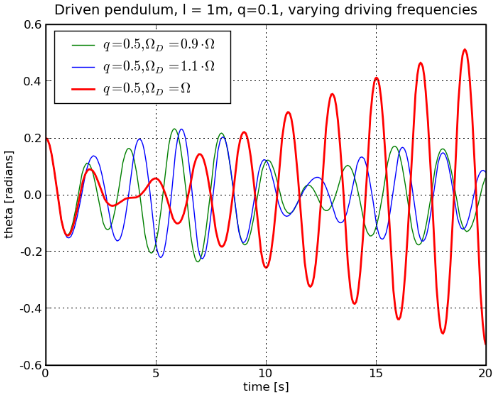


Driven oscillation: resonance

- $\Omega_D \to \Omega$ results in resonance

amplitude increases without bounds when no dissipation - driving force is in resonance with eigenfrequency

$$\theta_{\max} = \frac{F_D}{\sqrt{(\Omega^2 - \Omega_D^2)^2 + (q\Omega_D)^2}}$$



Real oscillator: adding non-linearity

- So far assumed amplitude is small $\sin(\theta) \to \theta$: not always a good approximation e.g resonance!
- For the description of a more realistic pendulum, we reinstate the non-linearity, and we will use $\sin\theta$ instead of making the small angle approximation
- This will have interesting consequences:
  - In the non-driven, non-dissipative pendulum, the eigen-frequency depends on the amplitude
  - Driving force leads to chaotic motion see next lecture

Summary

- Harmonic motion is a very important phenomenon in physics worthwhile to study in great detail. We focussed on a pendulum here but many other examples
- Euler's method fails to describe harmonic motion properly, due to non-conservation of energy. The Euler-Cromer method works much better.
- Adding dissipation and driving force adds new-phenomena: damping and resonances
- Adding non-linearity paves the road towards deterministic chaos, the subject of next lecture.
- In the homework assignment you'll be asked to implement a full simulation of the pendulum in the Euler-Cromer method, including dissipation, driving force, and non-linearity.