6. Compressible Fluids.

Sound Waves. Consider waves of small amplitude, propagating in a compressible, non-viscous medium. Assume motion to be adiabatic ($p \propto \rho^{\gamma}$) since p and ρ can change on shorter timescales than heat can conduct. Apply perturbation theory (i.e. consider small, linear departures from p_0, ρ_0 and $\underline{u}$=0.)

Let $\rho = \rho_0 + \delta\rho = \rho_0(1+s)$ where the density contrast, $s = \dfrac{\delta\rho}{\rho_0} << 1$.

Equation of Continuity. $\dfrac{\partial\rho}{\partial t} + \nabla.(\rho\underline{u}) = 0 \quad \therefore \dfrac{\partial}{\partial t}(1+s) + \nabla.[(1+s)\underline{u}] = 0$

Taking leading terms:- $\underline{\dfrac{\partial s}{\partial t} + \nabla.\underline{u} = 0}$ (1)

Euler Equation. $\dfrac{\partial \underline{u}}{\partial t} + (\underline{u}.\nabla)\underline{u} = -\dfrac{1}{\rho}\nabla p$

Linearise, assuming $\underline{u}$ of order s. $\therefore \dfrac{\partial \underline{u}}{\partial t} = -\dfrac{1}{\rho_0}\nabla p$ (2)

Since adiabatic, $p=p(\rho)$:- $\therefore \nabla p = \dfrac{dp}{d\rho}\nabla\rho$, $\left[\dfrac{\partial p}{\partial x} = \dfrac{dp}{d\rho}\dfrac{\partial\rho}{\partial x}\right]$

$$\sim \rho_0 \left.\frac{dp}{d\rho}\right|_0 \nabla s = \rho_0 c_0^2 \nabla s$$

where $c_0^2 = \left.\dfrac{dp}{d\rho}\right|_0 = \dfrac{\gamma p}{\rho} = \gamma\dfrac{R}{m}T$ (for adiabatic, ideal gas).

Speed of sound, $c_0 = 330ms^{-1}$, in still air at sea level.

Then substitute for ∇p in linearised Euler Equation (2)

$\rightarrow \quad \dfrac{\partial \underline{u}}{\partial t} + c_0^2\nabla s = 0$ (3)

Eliminate $\underline{u}$ between Euler and Continuity Equations by taking $\dfrac{\partial}{\partial t}$ of (1) and substitute for $\dfrac{\partial \underline{u}}{\partial t}$ from (3) :- $\therefore \dfrac{\partial^2 s}{\partial t^2} - c_0^2\nabla.\nabla s = 0$

i.e. $$\underline{\nabla^2 s - \frac{1}{c_0^2}\frac{\partial^2 s}{\partial t^2} = 0} \qquad (4)$$

(1-D: - $\frac{\partial^2 s}{\partial x^2} - \frac{1}{c_0^2}\frac{\partial^2 s}{\partial t^2} = 0$).

This has form of WAVE EQUATION for the density contrast. Assume irrotational motion and let ϕ^+ be velocity potential such that $\underline{u} = \nabla\phi^+$.

Substitute in linearised Euler equation (3) :-

$$\therefore \frac{\partial}{\partial t}\left(\nabla\phi^+\right) + c_0^2\nabla s = 0$$

$$\therefore \nabla\left(\frac{\partial\phi^+}{\partial t} + c_0^2 s\right) = 0$$

$\therefore \frac{\partial\phi^+}{\partial t} + c_0^2 s = f(t)$ $\qquad$ $\therefore \underline{s = -\frac{1}{c_0^2}\frac{\partial\phi}{\partial t}}$ $\qquad$ where $\phi = \phi^+ - \int f(t)dt$ i.e. f(t) has been absorbed in ϕ.

Subs. for s in continuity equation (1) :-

$\nabla.\underline{u} + \frac{\partial s}{\partial t} = 0 \qquad \rightarrow \qquad \underline{\nabla^2\phi - \frac{1}{c_0^2}\frac{\partial^2\phi}{\partial t^2} = 0}$ $\qquad$ [1D: $\frac{\partial^2\phi}{\partial x^2} - \frac{1}{c_0^2}\frac{\partial^2\phi}{\partial t^2} = 0$]

Thus equation for ϕ is same form as for s - WAVE EQUATION.

ϕ, p, ρ all obey wave equation solutions; when solutions for s or ϕ are found then:-

$\underline{p = \rho_0(1+s)}$ $\qquad$ [or $\underline{\rho = \rho_0\left(1 - \frac{1}{c_0^2}\frac{\partial\phi}{\partial t}\right)}$]

Then $p = p_0 + \frac{dp}{d\rho}\left(\rho - \rho_0\right)$ $\qquad$ by Taylor Expansion

$= \underline{p_0 + \rho_0 c_0^2 s}$ $\qquad$ [$= \underline{p_0 - \rho_0\frac{\partial\phi}{\partial t}}$]

Solution of 1-D Wave Equation.

$$\underline{\phi = F_1(x - c_0 t) + F_2(x + c_0 t)} \qquad \text{(General Solution)}$$

where F_1 and F_2 are arbitrary functions with form set by initial conditions. $F_1(x\text{-}c_0t)$ is wave with speed c_0 in positive x direction and $F_2(x+c_0t)$ is wave with speed c_0 in negative x direction.

Example Consider incident wave from x<0 impinging on a rigid wall at x=0. Then incident part of wave is $\phi_i = F_1(x - c_0 t)$. Reflected part is $\phi_r = F_2(x + c_0 t)$. Hence $\phi_{total}=\phi_i+\phi_r$.= $F_1(x - c_0 t) + F_2(x + c_0 t)$.

At x=0, u=0 $\quad \therefore \dfrac{\partial \phi}{\partial x} = 0$.

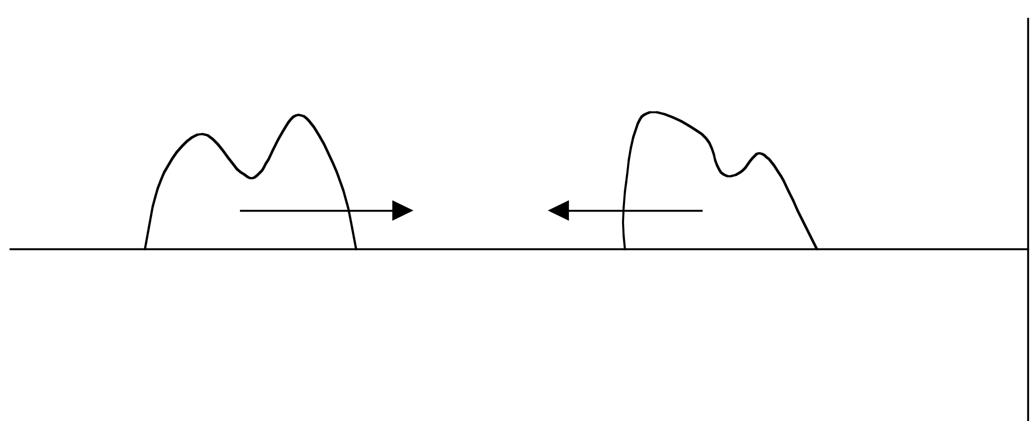
$\therefore F_1'(-c_0 t) + F_2'(+c_0 t) = 0$.

$\therefore F_2'(\alpha) = -F_1'(-\alpha)$.

$\therefore F_2(\alpha) = F_1(-\alpha)$ (integration includes division by –1).

$\therefore F_2(x + c_0 t) = F_1(-x - c_0 t)$

$\underline{\therefore \phi = F_1(x - c_0 t) + F_1(-x - c_0 t)}$.



Further Notes on Sound Waves.

Can also express solution for ϕ in Fourier harmonics:-

$$\phi = F_1(x - c_0 t) + F_2(x + c_0 t) = \underline{\sum_k A_k e^{ik(x-c_0 t)} + \sum_k B_k e^{ik(x+c_0 t)}}$$

Relating Density Contrast, s, and Velocity, u.

Harmonic of 1-D Sound Wave $\rightarrow \phi = \phi_0 \cos(kx - \omega t)$ where $\omega = c_0 k$.

Since $s = -\frac{1}{c_0^2}\frac{\partial \phi}{\partial t} \quad \rightarrow \quad s = -\frac{\omega}{c_0^2}\phi_0 \sin(kx - \omega t)$

But $\underline{u} = \nabla\phi = \underline{i}\frac{\partial \phi}{\partial x} \quad \therefore$ u = -k$\phi_0 \sin(kx - \omega t) = -\frac{\omega}{c_0}\phi_0 \sin(kx - \omega t)$

$\therefore \quad \underline{s = \frac{u}{c_0}}$ i.e. s << 1 $\Rightarrow \underline{u << c_0}$.

<u>i.e. perturbation in flow velocity caused by wave is << sound wave velocity.</u>

<u>Note :</u> $\delta\rho = \rho_0 s = \rho_0 \frac{u}{c_0}, \quad \delta p = \frac{\gamma p_0}{\rho_0}\delta\rho = \gamma p_0 \frac{u}{c_0}$.

$\therefore \delta p \propto \delta\rho \propto u$ all <u>in phase</u>, ϕ is $\frac{\pi}{2k}$ <u>out of phase</u>.

Solution of 3-D Wave Equation.

Harmonic: $\phi = A\exp\{i(\underline{k}.\underline{x} - \omega t)\}$ where $\omega = |k|c_0$

$\rightarrow$ Plane wave, direction of propagation, <u>k</u>.

Now $\underline{u} = \nabla\phi = Ai\underline{k}\exp i(\underline{k}.\underline{x} - \omega t)$

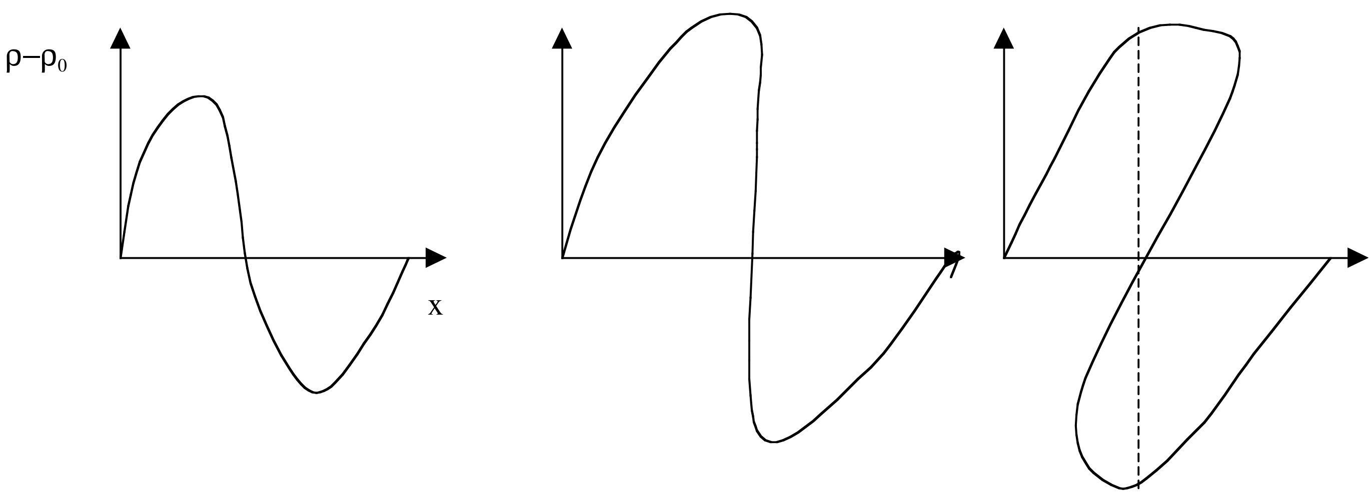
$\therefore$ Direction of motion of air is parallel to <u>k</u> $\rightarrow$ <u>longitudinal wave</u>.

Up to now we have assumed that density contrast, velocities are small. Otherwise the adiabatic sound speed is given by $c^2 = \frac{\gamma p}{\rho}$ (c.f. $c_0^2 = \frac{\gamma p_0}{\rho_0}$).

But $p = \frac{p_0}{\rho_0^\gamma}\rho^\gamma$ (adiabatic) . $\therefore c^2 = \frac{\gamma p_0}{\rho_0^\gamma}\rho^{\gamma-1}$.

As $\gamma = \frac{c_P}{c_V} > 1$ ($c_P - c_V = \frac{R}{m}$ where m is MW), it follows that c^2 increases with ρ.

Hence the <u>compression</u> part of a large amplitude sound wave moves faster than the <u>rarefaction</u> part and will eventually overtake it. In practice, a SHOCK WAVE forms.



<u>7. Entropy of a Fluid</u>.

The entropy per unit mass, S, is a characteristic function of the fluid. i.e. S is a function of only p, ρ, T. For a reversible process, $dS = \frac{dQ}{T}$. For irreversible processes, $dS > \frac{dQ}{T}$, even if dQ=0. (2nd Law of Thermodynamics).

Generally, we consider non-viscous, ideal fluid with no heat exchange between fluid elements i.e. adiabatic, dQ=0. If reversible processes only, ⇒S=constant for fluid element (≡ ISENTROPIC). If isentropic, flow S satisfies $\frac{DS}{Dt} = \frac{\partial S}{\partial t} + (\underline{u}.\nabla)S = 0$.
$\therefore$ for steady flow, entropy is constant alomg a streamline since $\underline{u}.\nabla S = 0$. When all fluid elements have the same constant value of S, the fluid is called HOMENTROPIC.
[includes all ADIABATIC flows considered so far.]

To show meaning of isentropic and homentropic.

Consider equation of state of ideal, compressible fluid with $\tau = \dfrac{1}{\rho}$ (τ=Volume/unit mass). $\underline{\therefore p\tau = \dfrac{R}{m}T}$

$$\therefore \frac{R}{m}dT = \tau dp + pd\tau \qquad (1)$$

1st Law of Thermodynamics applied to fluid element:

$$dQ = dE + pd\tau = c_V dT + pd\tau \qquad (2)$$

Eliminate dT between (1) and (2):

$$dT = \frac{dQ - pd\tau}{c_V} = \frac{\tau dp + pd\tau}{R/m}$$

$$\therefore c_V(\tau dp + pd\tau) = \frac{R}{m}(dQ - pd\tau)$$

$$\therefore c_V \frac{dp}{p} + \left(c_V + \frac{R}{m}\right)\frac{d\tau}{\tau} = \frac{R}{m}\frac{dQ}{p\tau} = \frac{dQ}{T} = dS$$ (assuming reversible+dividing through by $p\tau$)

$$\therefore c_V \frac{dp}{p} + c_P \frac{d\tau}{\tau} = dS$$ (since $c_P = c_V + \dfrac{R}{m}$)

Integrating, $\ln\dfrac{p}{p_0} + \dfrac{c_P}{c_V}\ln\dfrac{\tau}{\tau_0} = \dfrac{S - S_0}{c_V}$

$$\underline{\therefore \frac{p}{p_0}\left(\frac{\tau}{\tau_0}\right)^\gamma = \exp\left(\frac{S - S_0}{c_V}\right)}$$

then since $\tau = \dfrac{1}{\rho}$, $p = k\rho^\gamma$ where $k = \dfrac{p_0}{\rho_0^\gamma}\exp\left(\dfrac{S - S_0}{c_V}\right)$ is a function of the entropy, S.

For adiabatic, reversible conditions, S=constant, $\therefore$ k = constant.

When flow is isentropic, k has different value for each fluid element.

When flow is homentropic, k has same value for each fluid element.