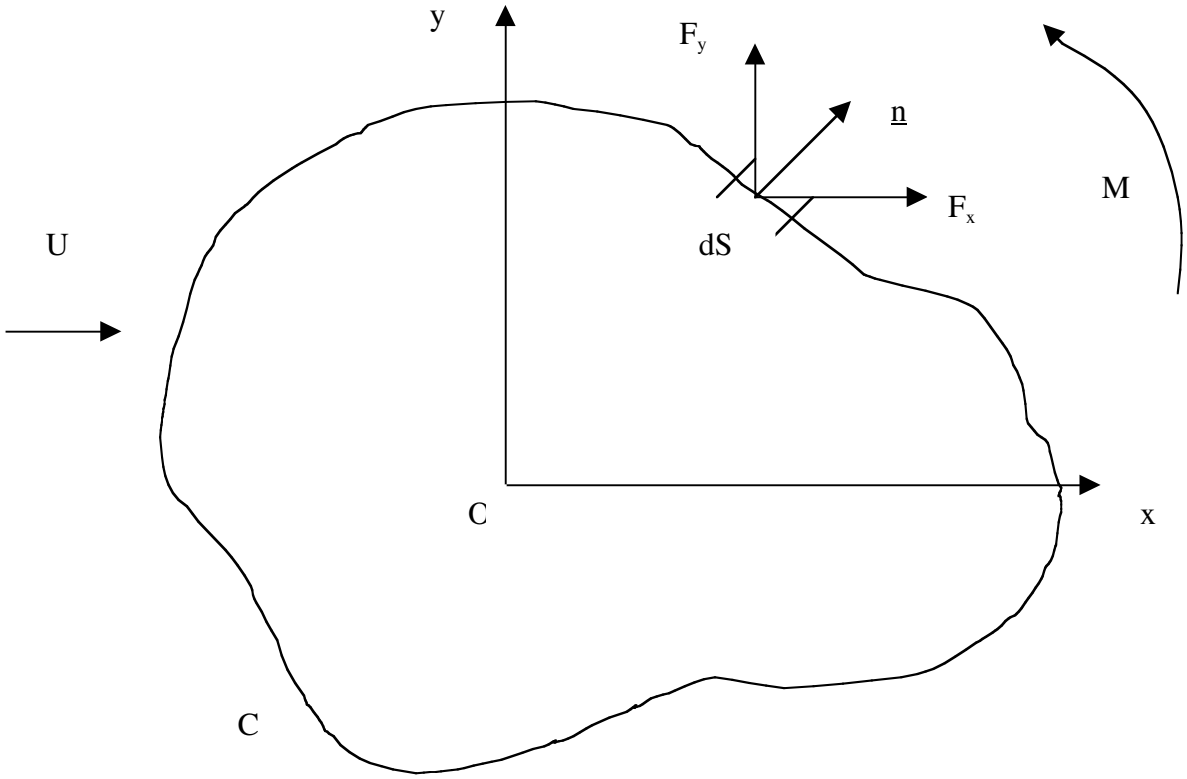
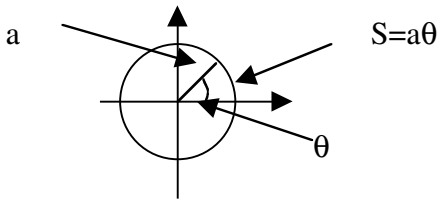
Blasius' Formulae for Force (and Couple) on C.




Force on dS = -p<u>n</u>dS

$n = \left(\frac{dy}{ds}, -\frac{dx}{ds}\right)$ is the normal, [ since tangent is $\left(\frac{dx}{ds}, \frac{dy}{ds}\right)$].

<u>Exercise</u>: Find <u>n</u> when C is a circle.




$\therefore$ Force on ds $= -p\left(\frac{dy}{ds}, -\frac{dx}{ds}\right)ds = \left(-pdy, pdx\right).$

Moment about O $= F_y x - F_x y = xpdx + ypdy.$

So if force is *(X,Y)* and moment is M:-

$$(X,Y) = \oint_C (-pdy, pdx)$$

$$M = \oint_C p(xdx + ydy)$$

$\therefore$ Forming *(X-iY)*: $X - iY = \int_C -pdy - ipdx = -i\int p(dx - idy) = \underline{\underline{-i\oint_C pdz^*}}$.

$$\therefore M = \underline{\underline{\mathrm{Re}\oint_C pzdz^*}} \qquad \left( = \mathrm{Re}\oint_C p(x+iy)(dx-idy) = \oint_C pxdx + pydy \right)$$

Steady Flow

Bernoulli $\rightarrow$ $p + \frac{1}{2}\rho_0\left|\frac{dw}{dz}\right|^2 = p_s \quad \rightarrow \quad p - p_s = -\frac{1}{2}\rho_0\left|\frac{dw}{dz}\right|^2$.

$$\therefore X - iY = i\frac{\rho_0}{2}\oint_C \left|\frac{dw}{dz}\right|^2 dz^*$$ (absorbing $p_s$ into p).

$$\therefore M = -\frac{1}{2}\rho_0 \,\mathrm{Re}\oint z\left|\frac{dw}{dz}\right|^2 dz^*$$

Now $\left|\frac{dw}{dz}\right|^2 = \frac{dw}{dz}\left(\frac{dw}{dz}\right)^* = \frac{dw}{dz}\frac{dw^*}{dz^*}$ (check!)

$$\therefore dz^*\left|\frac{dw}{dz}\right|^2 = \frac{dw}{dz}dw^* \qquad \therefore dw^* = d\phi - id\psi = d\phi = \mathrm{dw}$$ (since on C, $d\psi=0$).

$$\therefore X - iY = i\frac{\rho_0}{2}\oint \frac{dw}{dz}dw = i\frac{\rho_0}{2}\oint\left|\frac{dw}{dz}\right|^2 dz$$ Blasius' Formulae

$$\therefore M = -\frac{1}{2}\rho_0 \,\mathrm{Re}\oint\left(\frac{dw}{dz}\right)^2 zdz$$ - evaluate using Cauchy's Theorem

Force (General Case)

From above:

$$X - iY = i\frac{\rho_0}{2}\oint\left(U - \frac{i\Gamma}{2\pi z} + \frac{A_1}{z^2} + \frac{A_2}{z^3} + ....\right)^2 dz$$

$$= i\frac{\rho_0}{2}\oint\left(U^2 - \frac{2i\Gamma U}{2\pi z} + \frac{a_1}{z^2} + \frac{a_2}{z^3} + ....\right)dz$$

$$= 2\pi i\left(\frac{1}{2}i\rho_0 . \frac{-i\Gamma U}{\pi}\right) = \underline{\underline{i\rho_0 U\Gamma}}$$

by Cauchy's Theorem of Residues.

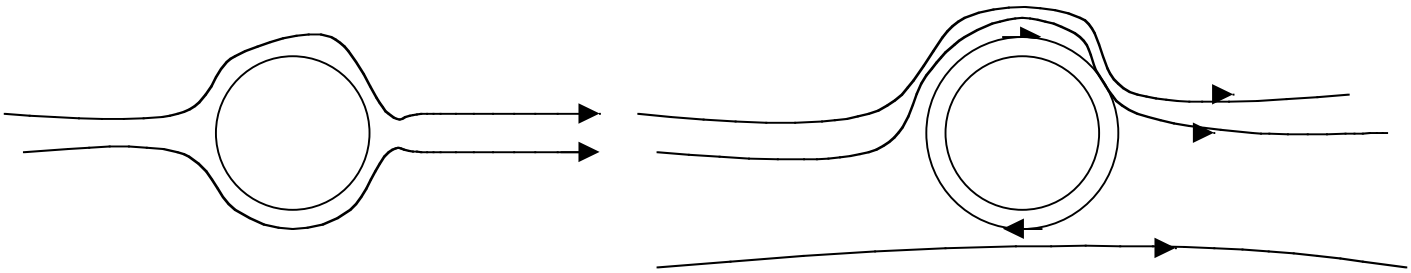
So $\underline{X(\parallel U) = 0}$ (no "drag") (not realistic unless streamlined and/or circulation).

$\underline{Y(\parallel U) = -\rho_0 U\Gamma}$ $\rightarrow$ "lift". (sign of lift depends on sign of circulation).

Could repeat exercise for M but left with $a_1$ which depends on form of C.

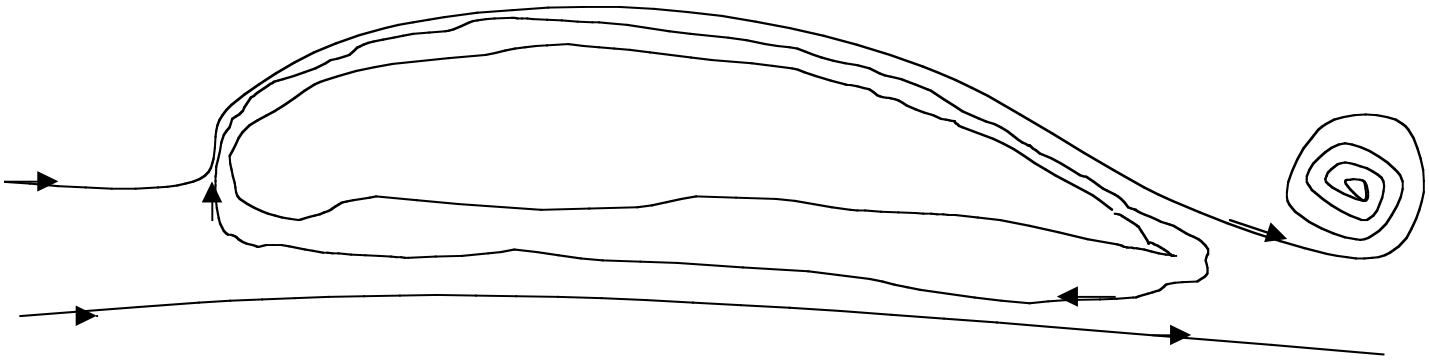
Effects of Circulation.

1. Circulation removes rear stagnation point which keeps inviscid flow solution "realistic" $\Rightarrow$ low drag.



2. On underside, low velocities mean high pressure (Bernoulli) which gives lift.

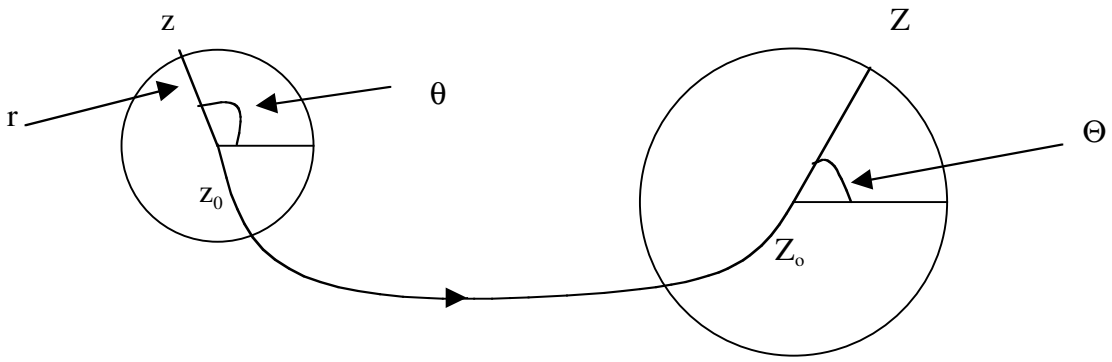
3. For case of cylinder, rotation of cylinder can produce circulation. Aerofoil shape induces circulation automatically with the same benefits.



Conformal Mapping.

Can regard regular function *f(z)* as a mapping $z \to Z$ where $Z=f(z)$. Such mappings are conformal at all points except when $f'(z)=0$ (exceptional points). Conformal means that angles, senses and curve smoothness are preserved.

Proof.




Suppose $z_o \to Z_0$ ie $Z_0=f(z_0)$.

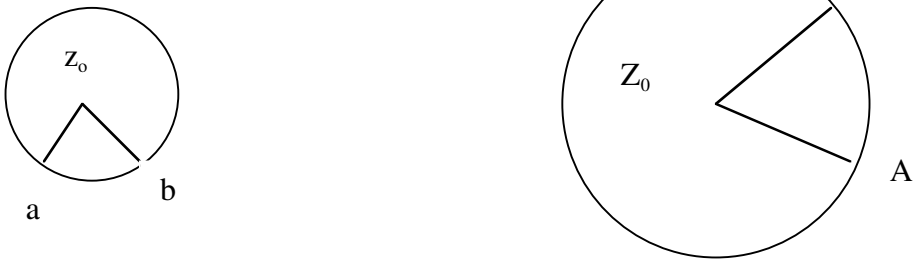
Expanding near $z_o$: $Z=f(z)=f(z_0)+(z-z_0)f'(z_0)$.

Express $Z-Z_0=Re^{i\Theta}$ and $z-z_0=re^{i\theta}$.

Take modulus: $R=r\left|f'(z_0)\right| \to$ isotropic magnification.

Since $f'(z_0)=\left|f'(z_0)\right|e^{i\arg f'(z_0)}$ then $\Theta=\theta+\arg f'(z_0)$.

$\therefore$ Direction not preserved but angle between two lines meeting at $z_0$ is preserved.




$\therefore$ If $f'(z_0)\neq 0$ on $z \in c$, then c smooth $\Rightarrow$ C smooth, c sharp $\Rightarrow$ C sharp, c simple, closed $\Rightarrow$ C simple, closed. Interior of c $\to$ Interior of C ie conformal mapping.

[ If $f'(z_0)=0,\ldots..$ but $f^n(z_0)\neq 0$, $Re^{i\Theta}=\frac{1}{n!}r^n e^{in\theta}f^n(z_0)$. $\therefore$ $\Theta=n\theta+\arg\left(f^n(z_0)\right) \Rightarrow$ angle not preserved at exceptional points.}

Joukowski transformation. $z = Z + \frac{A^2}{Z}$, A real and positive.

Takes ellipses (z) into circles (Z).

Exceptional points for Z→z are defined by :- $\frac{dz}{dZ} = 0 \Rightarrow 1 - \frac{A^2}{Z^2} = 0 \Rightarrow Z = \pm A$

Transform not regular at Z=0.

Take C to be $|Z| = R$, circle centre O, radius R.

$\therefore Z = \mathrm{R}e^{i\Theta} \quad 0 \le \Theta \le 2\pi$

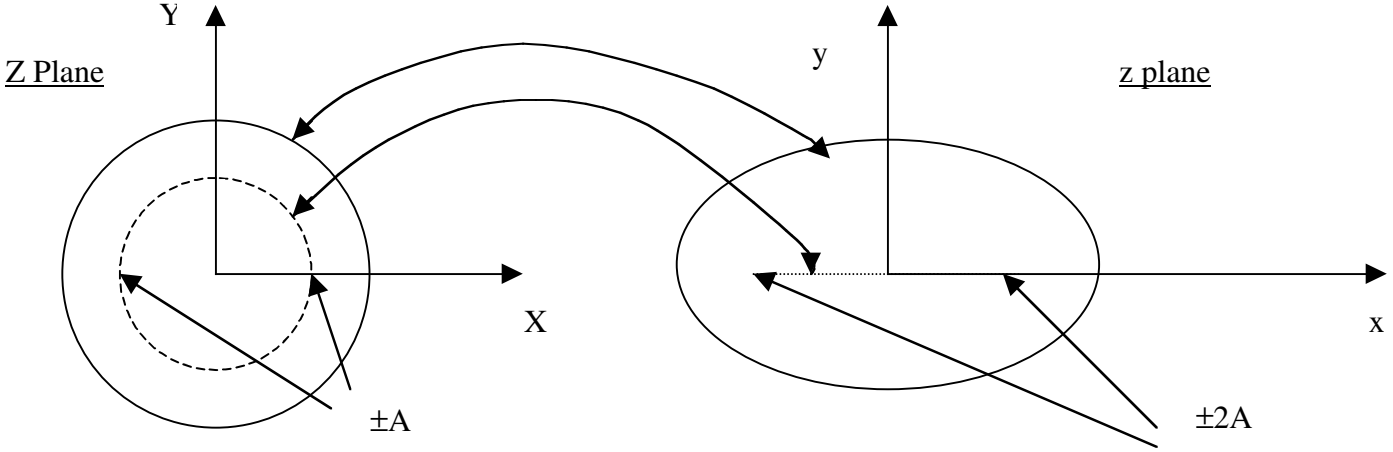
$\therefore z = R(\cos\Theta + i\sin\Theta) + \frac{A^2}{R}(\cos\Theta - i\sin\Theta)$

$\therefore x = \left(R + \frac{A^2}{R}\right)\cos\Theta, \qquad y = \left(R - \frac{A^2}{R}\right)\sin\Theta$

R>A → ellipse

semi-major axis: $R + \frac{A^2}{R} = a$, semi-minor axis: $R - \frac{a^2}{R} = b \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

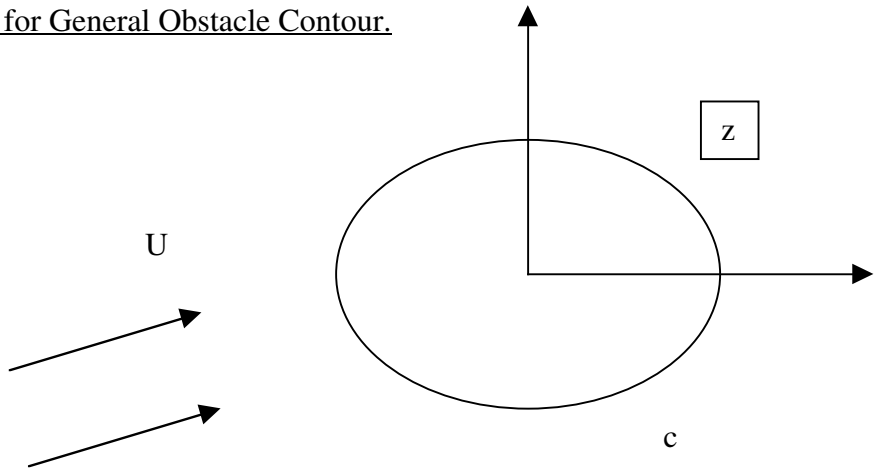
N.B. $\frac{1}{2}(a+b) = R, \qquad \frac{1}{2}(a-b) = \frac{a^2}{R}$



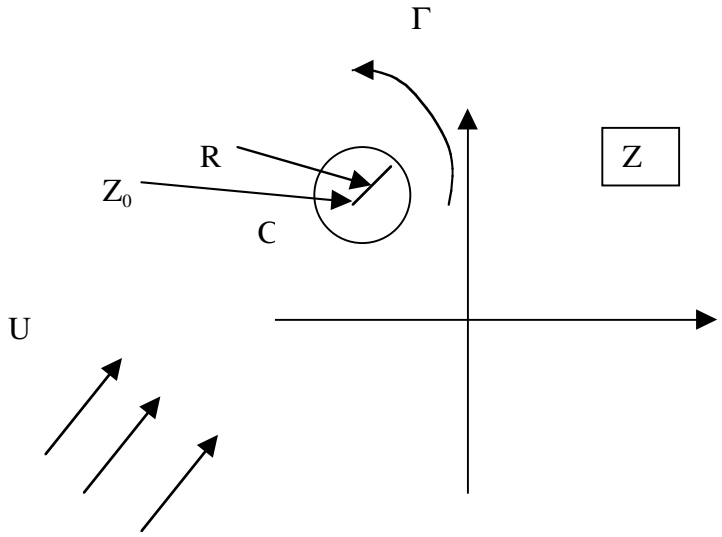

R=A → $x = 2A\cos\Theta, \quad y = 0 \rightarrow$ strip between ±2A.

So circle through exceptional points in Z plane → strip in z plane, $-2A \le x \le 2A$, y=0.

To Find w(z) for General Obstacle Contour.




Given $W_O(Z) = U\left[(Z - Z_0)e^{-i\alpha} + \frac{R^2 e^{i\alpha}}{Z - Z_0}\right] - \frac{i\Gamma}{2\pi}\ln\left[(Z - Z_0)e^{i\alpha}\right]$ corresponds to:-




Set $w(z) = W_O(Z(z))$ where Z(z) is regular map of c onto C since:

1. As $z \to \infty, \quad Z \to z$ and $w(z) \approx W_O(z)$.

2. When z lies on c, Z lies on C. $\mathrm{Im}(W_O(Z)) = 0$ when Z lies on C.
$\therefore \mathrm{Im}(w) = \mathrm{Im}(W_O(Z(z))) = 0$ when z lies on c ie c is a streamline.

3. $\frac{dw}{dz}$ is regular outside c because $\frac{dW}{dZ}$ is regular outside C $\left(\frac{dw}{dz} = \frac{dW}{dz} = \frac{dW}{dZ}\frac{dZ}{dz}\right)$.