6. $w(z) = U\left(z + \frac{a^2}{z}\right)$ U, a real.

w(z) analytic for z≠0. At $|z| >> a$, w(z) ~ Uz.

∴ Flow is uniform at large distances.

If $z = ae^{i\theta}$

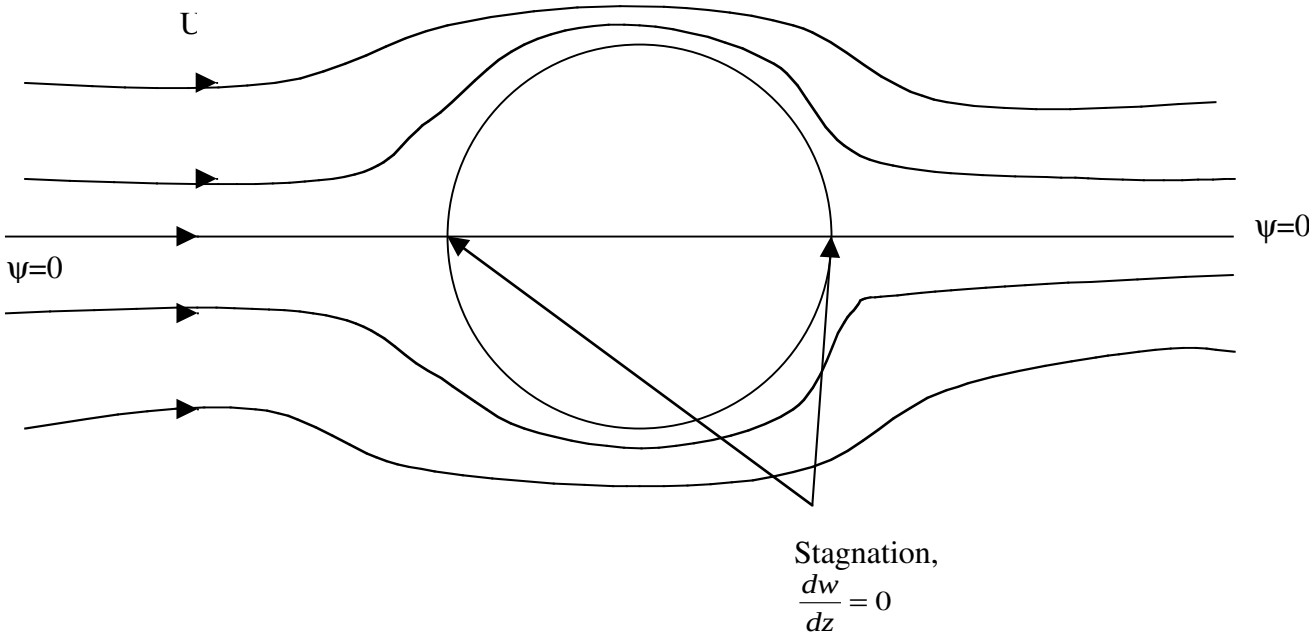
$\rightarrow \phi + i\psi = U\left(ae^{i\theta} + ae^{-i\theta}\right) = 2Ua\cos\theta$

$\therefore \psi = 0$ on $|z| = a$.

Generally, $w = U\left(re^{i\theta} + \frac{a^2}{r}e^{-i\theta}\right)$ $(z = re^{i\theta})$

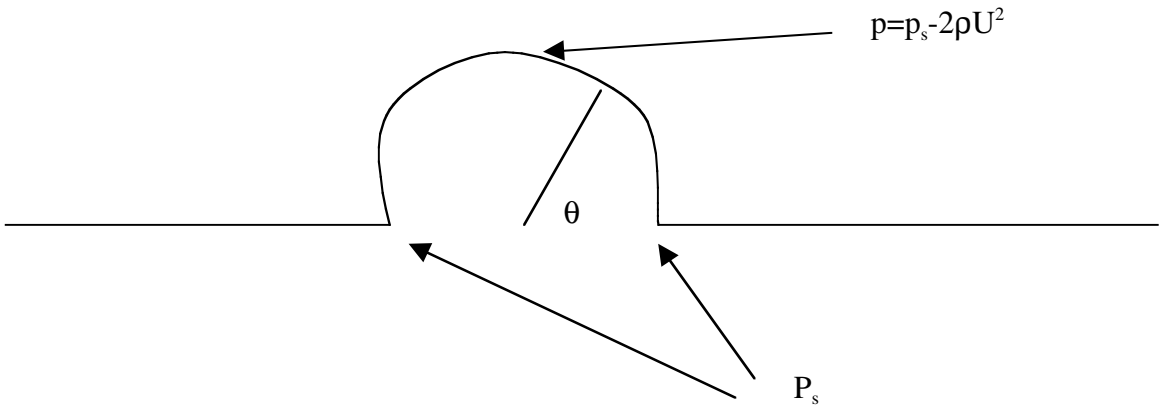
$\therefore \psi = U\sin\theta\left(r - \frac{a^2}{r}\right)$ - equation of streamlines

→Circle in a unifrom stream, $\frac{dw}{dz} = U\left(1 - \frac{a^2}{z^2}\right) \Rightarrow$ stagnation at z=±a.



Properties.

Bernoulli→ $\frac{p}{\rho_0}+\frac{1}{2}q^2=\text{constant}$ [irrotational, No body force, $\frac{\partial\phi}{\partial t}=0$.]



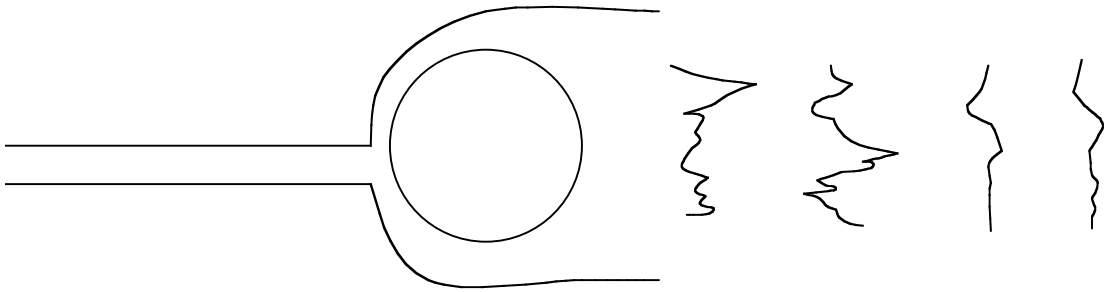
∴ Pressure at boundary in terms of stagnation pressure, p_s

$\frac{p}{\rho_0}+\frac{1}{2}q^2=\frac{p_s}{\rho_0}$ [q=0 at stagnation point].

On circle $q=\frac{1}{a}\frac{\partial\phi}{\partial\theta}=-2U\sin\theta$ $\left[\nabla\phi \text{ in cyl. polars, } \phi=2\text{Uacos}\theta\right]$

$\therefore \frac{p}{\rho_0}+\frac{1}{2}4U^2\sin^2\theta=\frac{p_s}{\rho_0}$

Pressure rises as s increases for $\frac{\pi}{2}>\theta>0$. ∴ Flow not realisable in a real fluid



Turbulent wake

Notes (1) Force on body = 0 from symmetry of body. In real case, no pressure recovery so real flow gives a force.

(2) w has singularity inside body at z=0.

(3) Flow not unique: $w=U\left(z+\frac{a^2}{z}\right)-\frac{i\Gamma}{2\pi}\ln\left(\frac{z}{a}\right)$ satisfies all conditions i.e. $w \sim Uz$ as $z\to\infty$, ψ=0 on boundary etc.

Contour Integration Revision.

Note: Analytic $\Rightarrow$ finite number of singularities, regular $\Rightarrow$ no singularities.

1. If f(z) is regular in annulus, A, $r_1 < |z_1 - z_2| < r_2$, then f(z) possesses in A a LAURENT EXPANSION, $f(z) = \sum_{-\infty}^{+\infty} a_n (z - z_0)^n$ with a_n complex constants.

2. If f(z) is regular in a punctured area $0 < |z - z_0| < \delta$ then f(z) has an isolated singularity at z_0. If, in Laurent Expansion, $a_{-p} \neq 0$ but $a_n = 0$ for $n < -p$ we say that f(z) has a pole of order p at z_0.

3. If f(z) has an isolated singularity at z_0 then the series of negative powers $\sum_{-\infty}^{-1} a_n (z - z_0)^n$ is called the principal part of f(z) at z_0. The coefficient, a_{-1}, is called the residue at z_0.

4. Cauchy's Theorem of Residues.

If f(z) is regular except for poles in domain D and if C is a closed path lying in D, then

$\oint_C f(z) dz = 2\pi i \sum_j r_j$ where r_j is a residue of f(z) at pole z_j inside C.

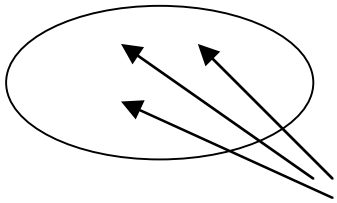
Summarising contour integration less mathematically:

1. If f(z) is regular in annulus $\Rightarrow$ f(z) has Laurent Expansion.

2. If $(z-z_0)^{-p}$ is biggest negative power in Laurent Expansion then f(z) has a pole of order p at z_0.

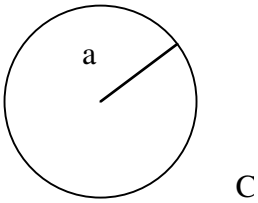
3. a_{-1} coefficient is called residue at z_o.

4. If f(z) regular except for poles then $\oint_C f(z)dz = 2\pi i \sum_j r_j$ where r_j is residue of pole j inside C.



Example to illustrate Cauchy's Theorem of Residues.

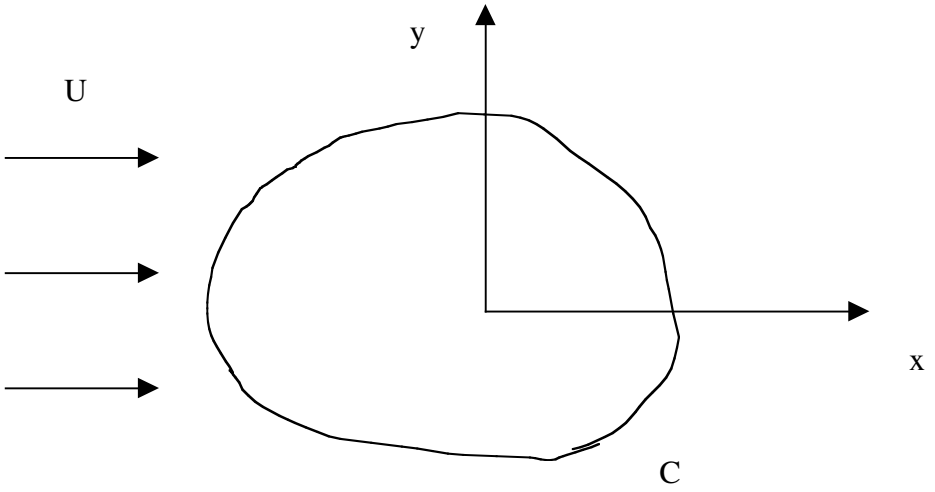
Integrate $f(z)=z^{-n}$ when C is a circle, radius, a, around origin.



$$\underline{n \neq 1} \rightarrow \quad \int_C z^{-n} dz = \frac{1}{1-n}[z^{1-n}] = \frac{1}{1-n}\left[a^{1-n}e^{(1-n)i\theta}\right]_0^{2\pi} = \underline{0} \qquad (z = ae^{i\theta})$$

$$\underline{n = 1} \rightarrow \quad \int_C \frac{1}{z}dz = [\ln z] = [\ln a + i\theta]_0^{2\pi} = \underline{2\pi i}$$

Flow past an arbitrary obstacle.



Overall Aim: to find force and moment of rigid obstacle caused by stream.

First: find approximate complex potential, w(z), for this problem. Want analytic w(z) which satisfies following conditions:-

1. $w \sim Uz$ as $z \to \infty$.

2. *Im(w)=0* on C [ψ=0 on C is streamline].

3. *w* has no poles outside C $\rightarrow$ regular (i.e. no sources or vortex points outside C.)

For cases considered here, we also assume that $\frac{dw}{dz}$ is regular outside C.

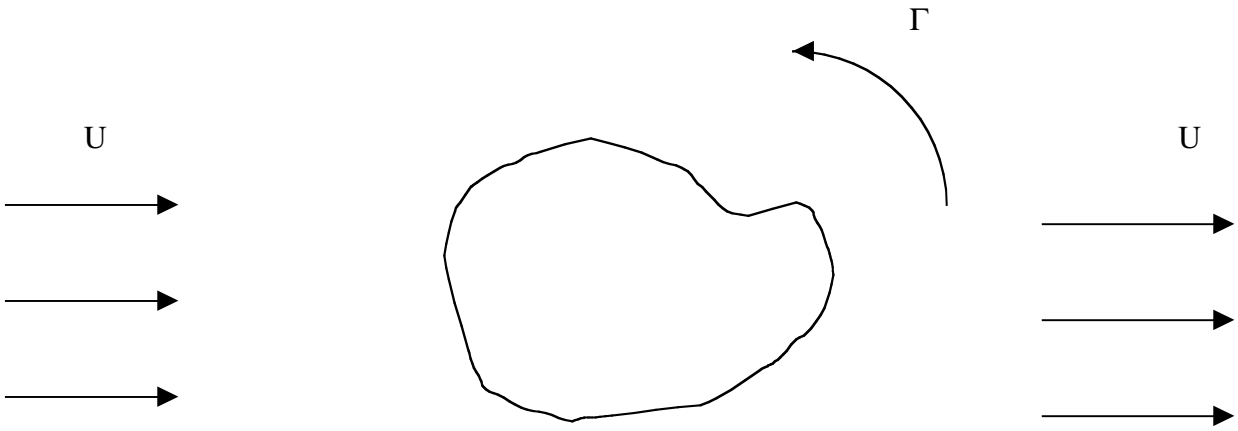
$\therefore \frac{dw}{dz} = U + \frac{A_0}{z} + \frac{A_1}{z^2} + \frac{A_2}{z^3} + \ldots$ [Laurent Expansion].

$\therefore\ w = Uz + A_0 \ln z - \frac{A_1}{z} - \frac{A_2}{2z^2} - \ldots.$ [Laurent Expansion since $\frac{dw}{dz}$ regular outside C].

Supppose $A_0 = \frac{1}{2\pi}(Q - i\Gamma)$ Q, Γ real.

$\therefore$ Q=0 (no sources in flow by assumption).

$\therefore w = Uz - \frac{i\Gamma}{2\pi} \ln z - \frac{A_1}{z} - \frac{A_2}{2z^2} - \ldots.$ (allowing line vortex).

i.e. flow looks like stream parallel to Ox plus a line vortex at large radii.

Γ - circulation to be <u>given</u> for case of arbitrary obstacle $\rightarrow$ see how to calculate for streamlined aerofoil shape later.