Irrotational Motion in the Plane of an Incompressible, Inviscid Flow.

Irrotational 2-D motion in x-y plane with velocity *(u(x,y,t), v(x,y,t), 0)*. Unit thickness in z direction.

Irrotational $\Rightarrow \nabla \times \underline{u} = \underline{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$

Incompressible $\Rightarrow \nabla . \underline{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ (Continuity Equation)

Therefore, following equations must be satisfied:-

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1)$$

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \quad (2)$$

Already used (2) (*curl* $\underline{u}$ *=0*) $\rightarrow$ velocity potential, φ, with $\underline{u} = \nabla\phi$ (i.e. $u = \frac{\partial \phi}{\partial x}, v = \frac{\partial \phi}{\partial y}$)

Then (1) $\rightarrow \underline{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \nabla^2 \phi = 0}$

[Laplace's Equation $\rightarrow$ unique solutions given $\underline{u}.\underline{n}=0$ on boundary]

[In 3-D, $\phi = -\frac{q}{r}$ is example of flow solution for spher. symm. flow $\rightarrow$ sources, sinks.]

[Euler/Bernoulli will determine p from u.]

Next, introduce stream function.

Stream function for 2-D Incompressible Flow.

In E-M, $\nabla . \underline{B} = 0$, implies there exists vector potential, $\underline{A}$, such that $\underline{B} = \nabla \times \underline{A}$

$(\nabla . (\nabla \times \underline{A}) = 0)$

Incompressible $\Rightarrow$ $\nabla.\underline{u}=0 \Rightarrow \underline{u}=curl\underline{A}$.

For 2-D flow need only specify $\underline{k}$ component of $\underline{A}$ since :-

$$\underline{u}=curl\underline{A}=\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}=\underline{i}\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)-\underline{j}\left(\frac{\partial A_3}{\partial x}-\frac{\partial A_1}{\partial z}\right)+\underline{k}\left(\frac{\partial A_2}{\partial x}-\frac{\partial A1}{\partial y}\right)$$

2-D $\Rightarrow$ no $\underline{k}$ component of $\underline{u}$ $\Rightarrow$ can take A_1, A_2=0.

Hence $\underline{A}=A_3\underline{k}=\psi(x,y)\underline{k}$.

ψ(x,y) is called the <u>Stream Function</u>: $u=\dfrac{\partial A_3}{\partial y}=\dfrac{\partial \psi}{\partial y}, v=-\dfrac{\partial A_3}{\partial x}=-\dfrac{\partial \psi}{\partial x}$.

Isocontours of ψ are <u>streamlines</u> since:-

Streamlines $\rightarrow \dfrac{dx}{u}=\dfrac{dy}{v} \therefore$ *vdx-udy=0*

$\dfrac{\partial \psi}{\partial x}dx+\dfrac{\partial \psi}{\partial y}dy=0 \Rightarrow d\psi=0$ along a streamline.

Thus instantaneous streamlines are $\psi(x,y,t)=f(t)$.

Steady Motion $\Rightarrow$ $\psi(x,y)=$ constant $\rightarrow$ streamlines.

If $u=\dfrac{\partial \psi}{\partial y}$ and $v=-\dfrac{\partial \psi}{\partial x}$ then equn,. (1) is automatically satisfied and problem reduces to solving (2) which becomes $\rightarrow \underline{\dfrac{\partial^2 \psi}{\partial x^2}+\dfrac{\partial^2 \psi}{\partial y^2}=\nabla^2\psi=0}$.

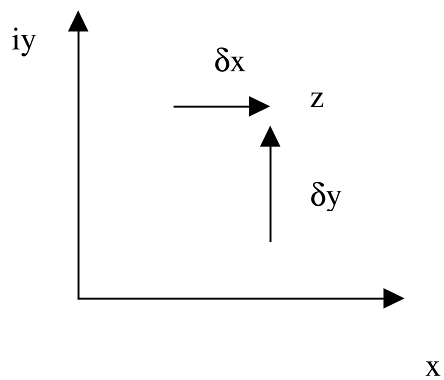
NB $\nabla\phi.\nabla\psi=\underline{u}.\nabla\psi=0$ since ψ=constant curves are streamlines and parallel to velocity.

Therefore ϕ and ψ are orthogonal functions.

Analytic Complex Functions.

Consider complex variable z=x+iy and the function w(z). This function is analytic (differentiable) if :-

$$\frac{dw}{dz} = \lim_{\delta z \to 0} \frac{w(z+\delta z) - w(z)}{\delta z} \text{ exists.}$$



For example, take $\delta z = \delta x$,

then $\frac{dw}{dz} = \frac{\partial w}{\partial x}$.

Otherwise if $\delta z = i\delta y$ then $\frac{dw}{dz} = \frac{\delta w}{i\delta y} = -i\frac{\partial w}{\partial y}$.

Setting $w = \phi + i\psi$ we have:-

$$\frac{\partial w}{\partial x} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x}$$
$$-i\frac{\partial w}{\partial y} = -i\frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial y}$$

These last two expressions for $\frac{dw}{dz}$ must be equal if *w* is an analytic function.

$\therefore \underline{\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}}$ and $\underline{-\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y}}$ (Cauchy- Riemann Equations)

<u>Complex Potential.</u>

If for some analytic function $w = \phi + i\psi$ we take:-

$u = \dfrac{\partial \phi}{\partial x}, v = \dfrac{\partial \phi}{\partial y}$ then (2) is automatically satisfied. (1) is also satisfied since forming:-

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x \partial y} = 0 \quad \text{(by Cauchy-Riemann).}$$

Hence any analytic *w(z)* forms a <u>complex potential</u> solution for a given 2-D incompressible flow with a particular set of boundary conditions. Complex potential *w(z)* contains in compact form all the information required to generate fluid velocities and streamlines.

Note: $\dfrac{dw}{dz} = \dfrac{\partial \phi}{\partial x} + i\dfrac{\partial \psi}{\partial x}$ (letting $\delta z \to \delta x$)

Then $\dfrac{dw}{dz} = u - iv$ $\quad (v = \dfrac{\partial \phi}{\partial y} = -\dfrac{\partial \psi}{\partial x})$ (by Cauchy-Riemann + Stream Function).

$\left|\dfrac{dw}{dz}\right|^2 = u^2 + v^2 = q^2$, where q is speed, $\left|\dfrac{dw}{dz}\right|$.

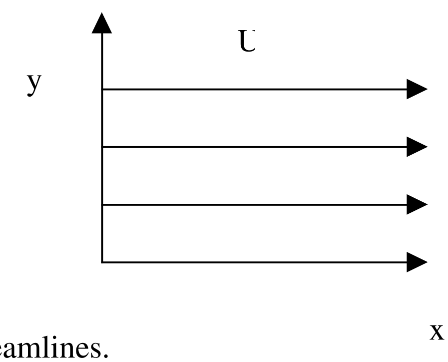
Any analytic w(z) represents an irrotational, incompressible 2-D flow.

<u>Examples</u>.

<u>1. $w=Uz = U(x+iy)$</u> $\quad (\Rightarrow \phi=Ux,\ \psi=Uy)$

$\dfrac{dw}{dz} = U$, u(x,y)=U, v(x,y)=0 $\qquad [\dfrac{dw}{dz} = u - iv]$

$\Rightarrow$ Uniform stream parallel to OX



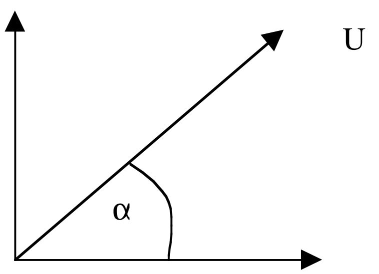
Alternatively $\underline{u} = \nabla\phi \to u = U, v = 0$.
Or ψ=*constant* $\Rightarrow$ lines y=*constant* are streamlines.

2. $\underline{w = Uze^{-i\alpha}}$, α constant.

$$\frac{dw}{dz} = Ue^{-i\alpha} = U\cos\alpha - iU\sin\alpha$$

$$\Rightarrow u = U\cos\alpha, v = U\sin\alpha$$

$\Rightarrow$



3. $\underline{w = \frac{Q}{2\pi}\ln z}$, Q real, constant , z≠0. (cut plane, $-\pi < \theta < \pi$, not differentiable at $\theta = \pi$)

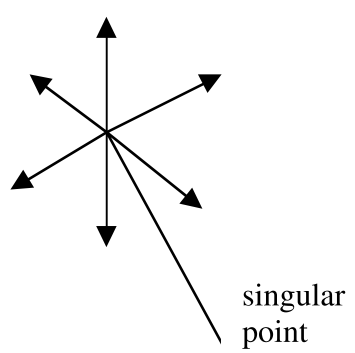
$$\phi = \mathrm{Re}\left(\frac{Q}{2\pi}[\ln r + i\theta]\right)$$

$$\therefore \phi = \frac{Q}{2\pi}\ln r, \quad \psi = \frac{Q\theta}{2\pi}$$

$$\therefore u_r = \nabla\phi = \frac{\partial\phi}{\partial r} = \frac{Q}{2\pi r}$$ (∇ in cylindrical polars)

$$u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = 0$$

i.e.



Line Source

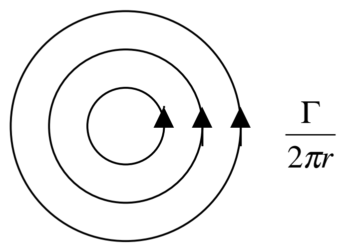
Flux across $|z| = R$, $\displaystyle\oint_{\substack{cylinder,\\ radius, R}} \underline{u}\,.\underline{dS} = \frac{2\pi RQ}{2\pi R} = Q$, independent of radius.

4. $\underline{w = -\frac{i}{2\pi}\Gamma \ln z}$ (Γ real) $z \neq 0$.

$$\phi = \frac{\Gamma\theta}{2\pi}, \quad \psi = -\frac{\Gamma}{2\pi}\ln r$$

$$u_r = \frac{\partial\phi}{\partial r} = 0, \quad u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{\Gamma}{2\pi r}$$

i.e.



Line Vortex

For circle radius R, $\oint_C \underline{u}.\underline{dX} = \int_0^{2\pi} u_\theta r d\theta = \int_0^{2\pi} \frac{\Gamma}{2\pi R} R d\theta = \Gamma$ (significance of Γ).

5. $\underline{w = -Cz^n}$ C real, >0, n constant.

$$\phi + i\psi = -C\left[re^{i\theta}\right]^n = -Cr^n e^{in\theta}$$

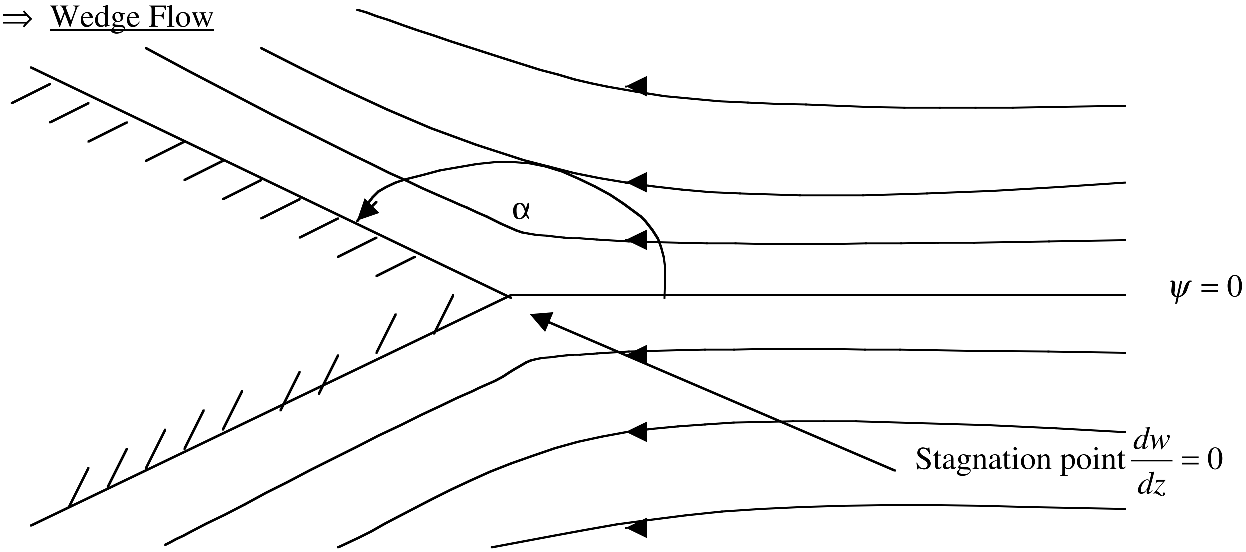
$$\phi = -Cr^n \cos n\theta, \qquad \psi = -Cr^n \sin n\theta$$

Now $\psi = 0$ on $\theta = 0,\ n\theta = \pi$

For $\underline{n > 1}$ then this $\theta \equiv \alpha = \dfrac{\pi}{n} < \pi$ also $\alpha = -\dfrac{\pi}{n}$.

$$\frac{dw}{dz} = -nCz^{n-1} \neq 0 \text{ if } z \neq 0$$
$$= 0 \text{ if } z \to 0.$$

$\Rightarrow$ Wedge Flow



Must have rising velocity/falling pressure along boundary for real flow (i.e. $\dfrac{\partial u}{\partial s} > 0, \quad \dfrac{\partial p}{\partial s} < 0 \quad$ as $s \to \infty$).

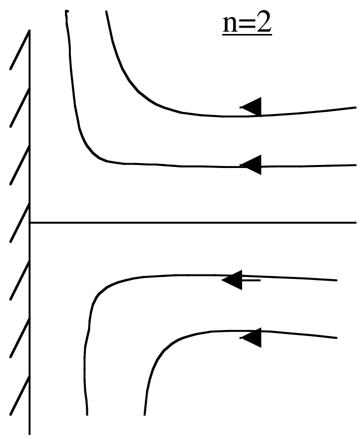
Now $\dfrac{\partial \phi}{\partial r} = -nCr^{n-1} \cos n\alpha = nCr^{n-1}$ (along boundary $n\alpha = \pi$).

Since $n > 1$, $u_r \to \infty$ as $r \to \infty$

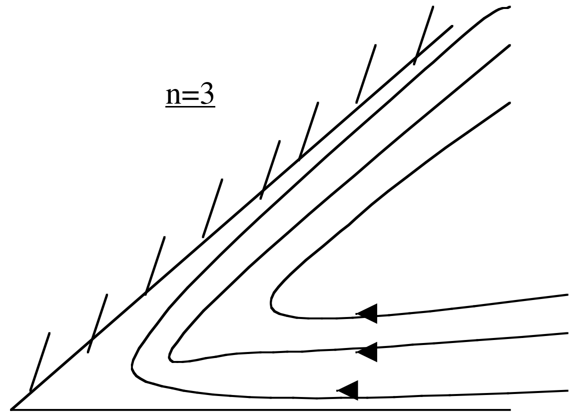
So $\dfrac{\partial u}{\partial s} > 0$ (ie $\dfrac{\partial p}{\partial s} < 0$) $\therefore$ Real Flow.

Special Cases

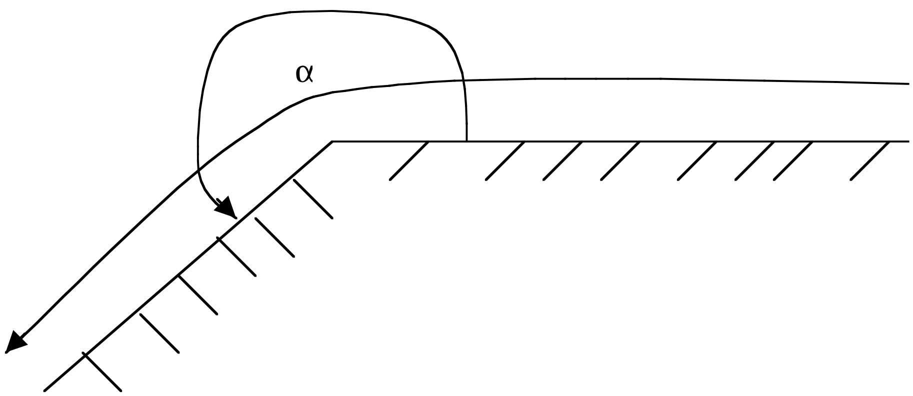
n=2



n=3



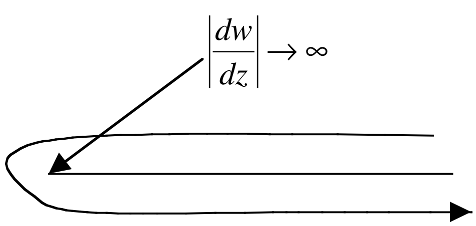
Next $\underline{n<1} \Rightarrow \underline{\alpha>\pi}$



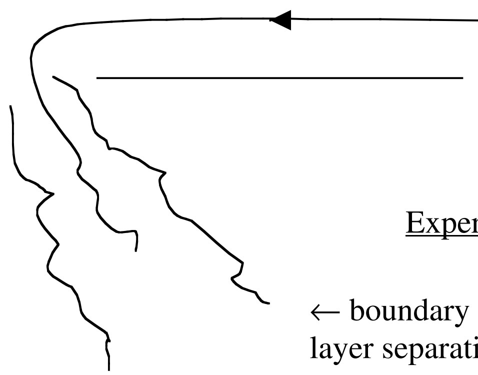
$$\frac{dw}{dz} = -nCz^{n-1} \quad \to \infty \text{ as } \quad z \to 0$$

$u_r \to 0$ as $r \to \infty$ $\quad \therefore \dfrac{\partial u}{\partial s} < 0 \to$ will not occur in real fluid.

n=1/2



Theory



Experiment

← boundary
layer separation