Helmholtz's Theorem.

If fluid inviscid and $p=f(\rho)$ only with $\underline{F} = -\nabla V$ then if motion starts from rest ($\underline{u}(\underline{X},0) = 0$ everywhere) then *curl* $\underline{u}=0$ always.

Proof In case of 2-D motion, incompressible fluid, (ρ=constant).

Euler Equations.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} - \frac{\partial V}{\partial x} \quad \} \quad \text{take } -\frac{\partial}{\partial y}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - \frac{\partial V}{\partial y} \quad \} \quad \text{take } \frac{\partial}{\partial x}$$

Now $$curl\underline{u} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & 0 \end{vmatrix} = \underline{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \underline{k}\xi$$

$$\therefore \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial y}\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = 0$$

2nd and 3rd terms in above equation:-

$$\rightarrow \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + u\frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + v\frac{\partial^2 v}{\partial x \partial y} - \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} - u\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} - v\frac{\partial^2 u}{\partial y^2}$$

$$= \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\frac{\partial u}{\partial y}$$

$$= \left(u\frac{\partial \xi}{\partial x} + v\frac{\partial \xi}{\partial y}\right) + \frac{\partial v}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \frac{\partial u}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

$$= u\frac{\partial \xi}{\partial x} + v\frac{\partial \xi}{\partial y} + 0 + 0$$

(by the Continuity Equation for Incompressible Fluid, *div* $\underline{u}=0$).

Hence combined Euler Equation of Motion reduces to:-

$$\frac{\partial \xi}{\partial t} + u\frac{\partial \xi}{\partial x} + v\frac{\partial \xi}{\partial y} = 0 \quad \text{where } \xi\underline{k} = curl\underline{u}$$

Suppose (x^P, y^P) are coords of fluid particle at time t.

Define $\xi^P = \xi(x^P, y^P, t)$

$$\therefore \frac{\partial \xi^P}{\partial t} + u\frac{\partial \xi^P}{\partial x} + v\frac{\partial \xi^P}{\partial y} = 0 \Rightarrow \frac{D\xi^P}{Dt} = 0$$

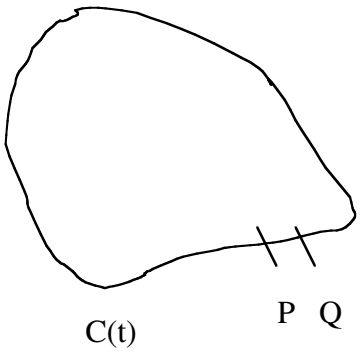
$\therefore \xi^P =$ constant for each fluid particle

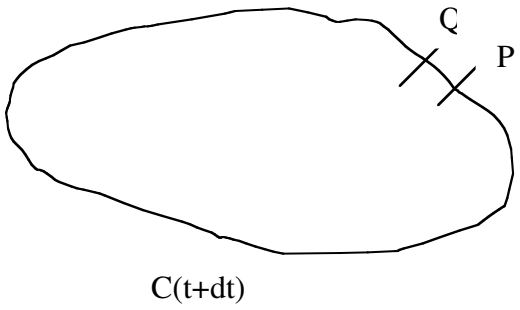
=0 since motion starts from rest.

$\therefore \xi = 0$ for all t i.e. *curl u=0* for all t.

Kelvin's Circulation Theorem.

For an inviscid fluid with p=f(ρ) only and $\underline{F} = -\nabla V$, if $\Gamma_C(t) = \int_C \underline{u}.\underline{dX}$ where C(t) is a closed circuit of fluid particles whose collective motion we are following, then $\frac{d\Gamma_C}{dt} = 0$.





(Proof of Kelvin's Theorem not given here.)

Note: Helmholtz/Kelvin $\Rightarrow$ if *curl u=0* everywhere at the beginning then *curl u=0* always. This inviscid fluid which starts irrotational remains irrotational. Explains importance of irrotational, inviscid flow.

4. Hydrostatics.

The static limit is attained where $\underline{u}=0$ everywhere for all t, in which case $\frac{D\underline{u}}{Dt}=0$. The equation of motion becomes:- $\nabla p=\rho\underline{F}$.

If the external forces are conservative then $\underline{F}=-\nabla V$ and so we have

$$\underline{\nabla p+\rho V=0} \quad \text{(Hydrostatic Euler Equation.)}$$

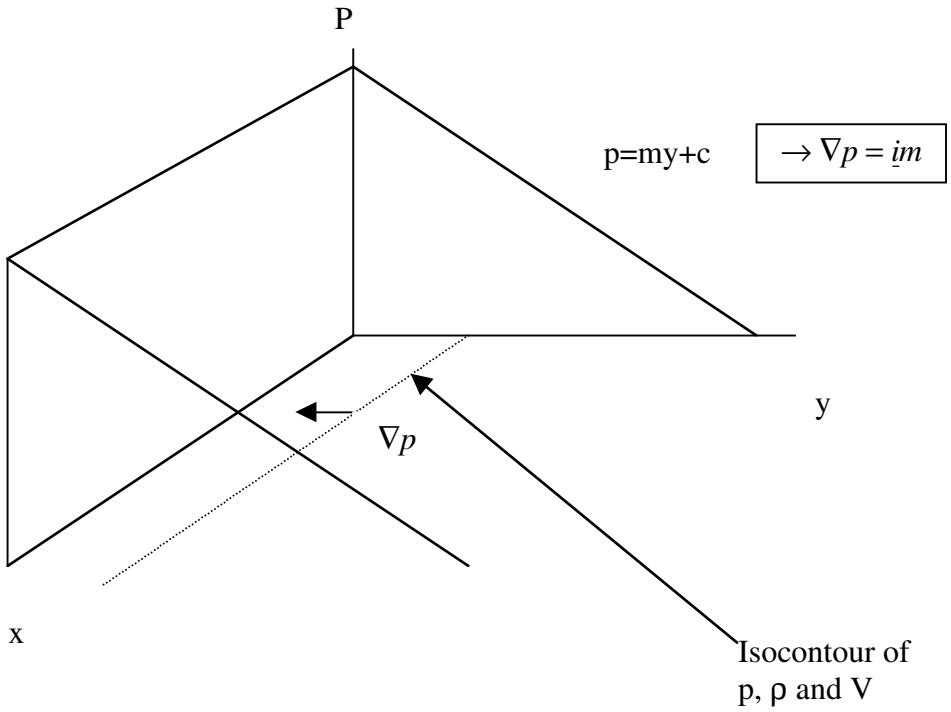
Operate with $\nabla\times$:-

$$\nabla\times(\nabla p)+\nabla\times(\rho\nabla V)$$
$$=\rho\nabla\times(\nabla V)+\nabla\rho\times\nabla V$$

$$\therefore \nabla\rho\times\nabla V=0$$

It follows that $\nabla\rho,\nabla p,\nabla V$ are all parallel vectors and hence that their isocontours coincide. Consequently, p=p(ρ) and V=V(ρ), a remarkable result.

Example 2-D



Exercise. From Hydrostatic Euler Equation, prove that $p = \rho g h + p_{At}$ at depth h below surface of incompressible liquid.

Earth's Atmosphere.

Potential corresponds to grav. field, $V = -\frac{C}{r}$ where C(=GM_E) is a constant.

In static equilibrium, Bernoulli $\rightarrow \int \frac{dp}{\rho} + V =$ constant (everywhere).

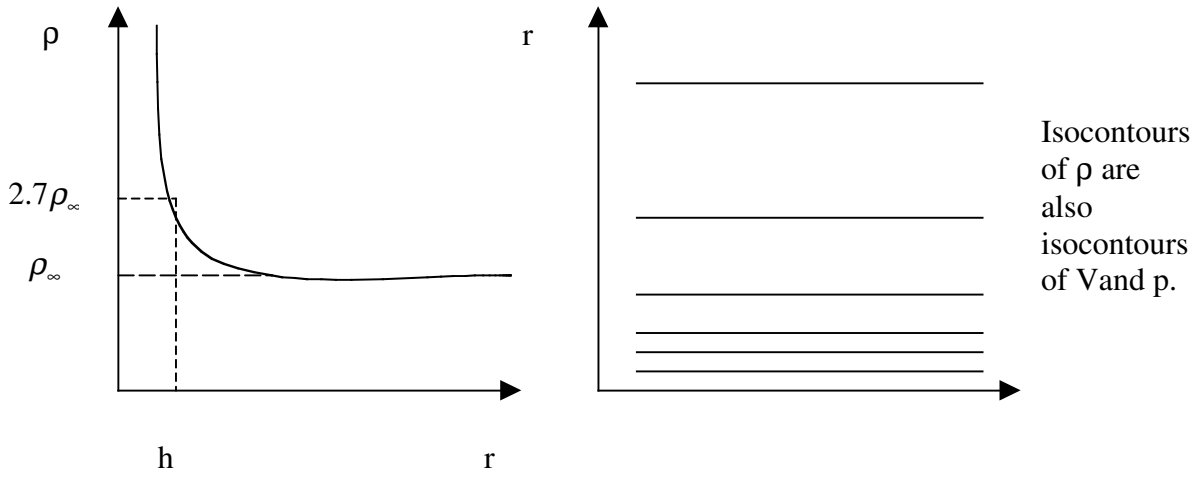
Suppose T= constant (isothermal atmosphere) Then from equation of state $dp = \frac{RT}{m} d\rho$.

Hence $\frac{RT}{m} \ln \rho - \frac{C}{r} =$ constant $=$ k.

$$\therefore \ln \rho = \frac{m}{RT}\left(k + \frac{C}{r}\right)$$

$$\therefore \rho = \exp\frac{mk}{RT} \exp\frac{mc}{RTr} = \rho_\infty \exp\frac{h}{r}$$

where $\rho_\infty = \rho(r = \infty)$ and $h = \frac{mc}{RT}$ (h is scale height of atmosphere).



Stellar Hydrostatic Equation.

Here $\underline{F} = -\nabla V$ where V is gravitational potential given by

$\nabla.\underline{F} = -4\pi G\rho$ or $\nabla^2 V = 4\pi G\rho$. (Poisson's Equation for $1/r^2$ law).

Now hydrostatic Euler $\rho\nabla V + \nabla p = 0$

$$\Rightarrow \nabla.\left(\frac{\nabla p}{\rho}\right) = -4\pi G\rho$$

Thus for spherically symmetric star:-

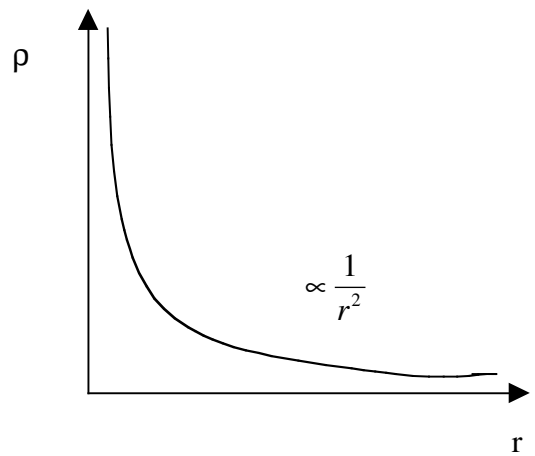
[since $\nabla.\underline{a} = \frac{1}{r^2 \sin^2\theta}\frac{\partial}{\partial r}\left(r^2 \sin\theta\, a_r\right)+\ldots\ldots..$]

$$\underline{\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dp}{dr}\right) = -4\pi G\rho}$$

For isothermal case, $p=C\rho$, where $C = \frac{R}{m}T$.

$$\therefore \underline{\frac{d}{dr}\left(\frac{r^2}{\rho}C\frac{d\rho}{dr}\right) = -4\pi G\rho r^2}$$

Check that $\rho = \frac{C}{2\pi G r^2}$ is a solution i.e. $\underline{\rho = \frac{\frac{R}{m}T}{2\pi G r^2}}$ (singular isothermal sphere).



5. Inviscid and Incompressible Flows.

Note on Reynolds Number.

It can be shown that viscous stress forces have magnitude

$F_\mu \approx \mu\nabla^2\underline{u} = \mu(\nabla^2 u, \nabla^2 v, \nabla^2 w)$ where μ is coefficient of viscosity ($ML^{-1}T^{-1}$).

(Stress $\rightarrow \mu\nabla\underline{u}$; $\nabla(\mu\nabla\underline{u})$ needed analagous to ∇p)

Assume U and L are the characteristic velocity and scale of the flow .

Net force on fluid element is $\rho\dfrac{D\underline{u}}{Dt} = \rho\left(\dfrac{\partial}{\partial t} + \underline{u}.\nabla\right)\underline{u}$ and $\underline{\rho(\underline{u}.\nabla)\underline{u} \sim \rho\dfrac{U^2}{L}}$.

Now define Reynolds Number:-

$$\text{R} = \frac{\text{magnitude of all forces}}{\text{magnitude of viscous forces}} = \frac{\dfrac{\rho U^2}{L}}{\dfrac{\mu U}{L^2}} = \frac{\rho}{\mu}UL = \underline{\frac{UL}{\nu}}$$

where $\nu = \dfrac{\mu}{\rho}$ $\rightarrow$ coefficient of kinematic viscosity.

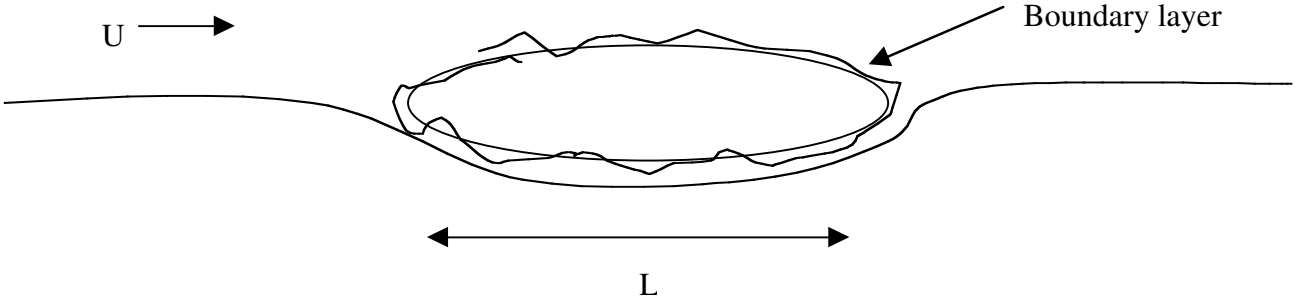
Viscous forces unimportant when $\underline{R >> 1}$.

Applicability of Inviscid Treatment to a Real Flow.

1. Fluid flow inviscid if dimensionless Reynplds number, R, is high (>>1).

2. Inviscid treatment good for free surface boundary conditions since fluid slips easily along such surfaces. Thus good for water waves, free jets, large gas bubbles etc.

3. Fluid sticks to solid surface. But if R>>1, viscous forces are confined to thin boundary layer provided pressure falls in flow direction ie $\dfrac{\partial p}{\partial s} < 0$ where s increases along streamline in direction of flow.



Proof: Bernoulli's Equation along streamlines of inviscid, incompressible fluid on steady flow with no body forces acting.

$$\frac{1}{2}u^2 + \frac{p}{\rho} = \text{ constant.}$$

Differentiating wrt s along streamline:-

$$\frac{\partial}{\partial s}\left(\frac{1}{2}u^2\right) + \frac{1}{\rho}\frac{\partial p}{\partial s} = 0.$$

$$\Rightarrow u\frac{\partial u}{\partial s} = -\frac{1}{\rho}\frac{\partial p}{\partial s} \Rightarrow \frac{\partial u}{\partial s} = -\frac{1}{\rho u}\frac{\partial p}{\partial s}$$

Thus, sign of $\frac{\partial p}{\partial s}$ important in boundary layer where u is lower than in external stream. If $\frac{\partial p}{\partial s}$ is negative, $\frac{\partial u}{\partial s}$ is positive which raises velocity in boundary layer and keeps it thin.
If $\frac{\partial p}{\partial s}$ is positive, boundary layer will thicken and make inviscid treatment invalid.