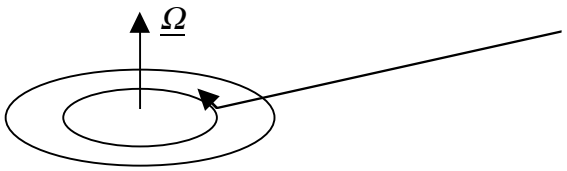
3. Concept of Vorticity.

curl $\underline{u}$=0 at some point $\underline{X}$, implies rotation of individual fluid particle at that point.

Solid Body Rotation – cylindrical polars



$$\underline{u} = (0, \Omega r, \theta)$$

$$curl\underline{u} = \frac{1}{r}\begin{vmatrix} \hat{\underline{r}} & r\hat{\underline{\theta}} & \hat{\underline{z}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ u_r & u_\theta & u_z \end{vmatrix}$$

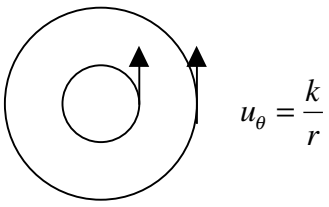
$$= \frac{1}{r}\begin{vmatrix} \hat{\underline{r}} & r\hat{\underline{\theta}} & \hat{\underline{z}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 0 & r\Omega r & 0 \end{vmatrix} = \hat{\underline{z}}\frac{1}{r}\frac{\partial}{\partial r}\left(\Omega r^2\right) = \hat{\underline{z}}2\Omega = (0,0,2\Omega).$$

Hence for solid body rotation, *curl* $\underline{u}$ =2x angular velocity vector everywhere in the "flow".

Curl $\underline{u}$ also called "vorticity".

Line vortex – cylindrical polars

$$u = \left(0, \frac{k}{r}, 0\right)$$

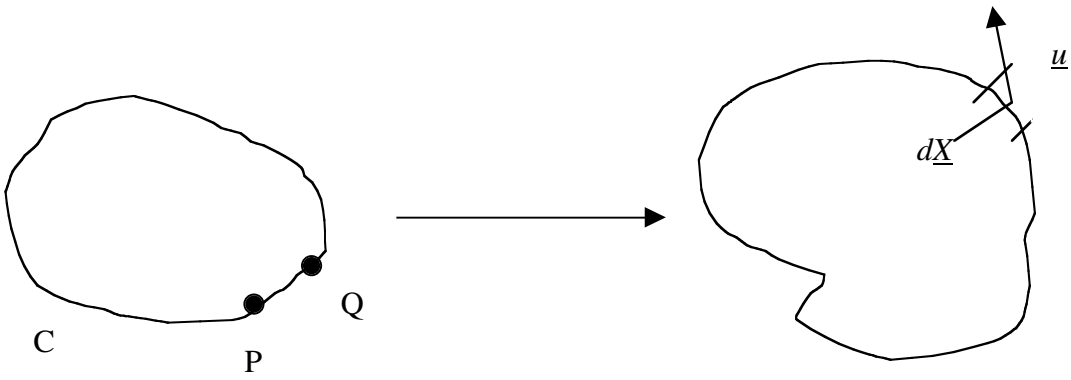


$$curl\underline{u} = \frac{1}{r}\begin{vmatrix} \hat{\underline{r}} & r\hat{\underline{\theta}} & \hat{\underline{z}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 0 & r\frac{k}{r} & 0 \end{vmatrix} = \hat{\underline{z}}\frac{0}{r} = 0 \text{ for } r \neq 0$$

So *curl* $\underline{u}$ = *0* everywhere except at *r=0* where vorticity is infinite. Irrotational flow means no rotation of fluid particles anywhere *ie curl* $\underline{u}$*=0* everywhere.

Circulation

Consider loop C of fluid particles – follow their motion:



Define $\Gamma_C = \oint_C \underline{u}.\underline{dX}$ $\rightarrow$ circulation – measure of total amount of rotation within the chosen loop.

Solid Body $\underline{u} = (0, \Omega r, 0)$

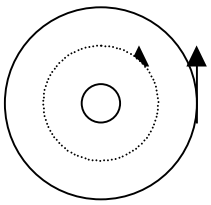
Choose loop C at radius r from axis of rotation

Here $dX = dr\hat{\underline{r}} + rd\theta\hat{\underline{\theta}}$ and so:-

$$\therefore \Gamma_C = \oint_C u_\theta r d\theta = \int_C \Omega r r d\theta = 2\pi r^2 \Omega$$

So circulation proportional to Ω and to area within loop, πr^2.

Line Vortex $u = \left(0, \frac{k}{r}, 0\right)$



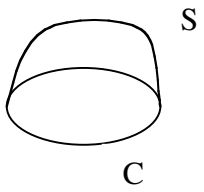
$u_\theta = \frac{k}{r}$

Choose loop C at radius r from rotation axis.

$$\Gamma_C = \int_C \underline{u}.\underline{dX} = \int \frac{k}{r} r d\theta = 2\pi k$$

(Circulation measures vortex strength.)

Recall Stokes' theorem $\int_S curl\underline{u}.\underline{dS} = \oint_C \underline{u}.\underline{dX}$



Since for line vortex, curl u =0, everywhere except centre, curves which don't enclose center have $\Gamma_C = 0$. Only curves containing axis of rotation have $\Gamma_C \neq 0$.

Example (Bernoulli + Line Vortex). Find pressure gradient in horizontal plane through a vertical tornado. Assume air incompressible.

Assume line vortex $\rightarrow$ irrotational flow.

Assume atmospheric conditions at $r = \infty$.

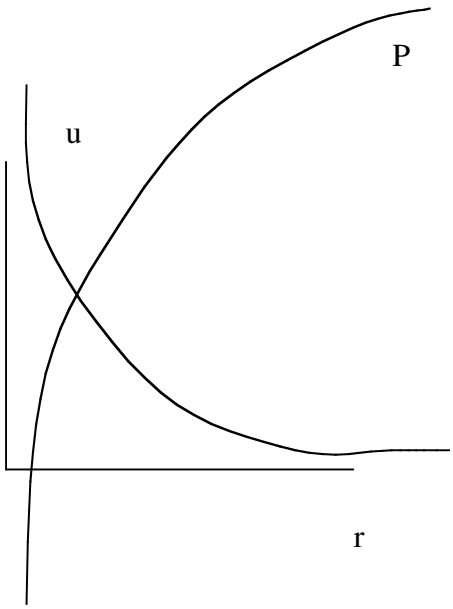
Bernoulli $\rightarrow$ $\frac{p}{\rho} + \frac{1}{2}u^2 + gz = \frac{p_{At}}{\rho} + \frac{1}{2}u_\infty{}^2 + gz_\infty$

Now if $u_\infty = 0$ and $z = z_\infty$ then

$$\frac{p}{\rho} + \frac{1}{2}u^2 = \frac{p_{At}}{\rho}$$

$$\therefore p - p_{At} = -\frac{1}{2}\rho u^2 = -\frac{1}{2}\rho\frac{k^2}{r^2}$$

$$\therefore p = p_{At} - \frac{1}{2}\rho\frac{k^2}{r^2}$$



Note curl $\underline{u}$ represents rotation fo fluid particle.

div $\underline{u}$ represents net outward flux per unit volume

Therefore, div $\underline{u}$ represents distortion of fluid particle.

In fact div u and curl u are anti-symmetric and symmetric parts of rate of strain tensor $\frac{\partial v_i}{\partial x_j}$ (see Paterson Chap. 5.1)

If fluid particle rotation develops $\Rightarrow$ existence of shear stress ie tangential force.

However, no shear stress in inviscid fluid.