Note Equation of Motion is $\frac{D\underline{u}}{Dt} = \frac{\partial \underline{u}}{\partial t} + (\underline{u}.\nabla)\underline{u} = \underline{F} - \frac{1}{\rho}\nabla p$.

Non-linear equation in $\underline{u}$ – can't solve by superposition methods – makes fluid mechanics hard!

Note that in components:-

$$\frac{D\underline{u}}{Dt} = \left\{ \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}, \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}, \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \right\}$$

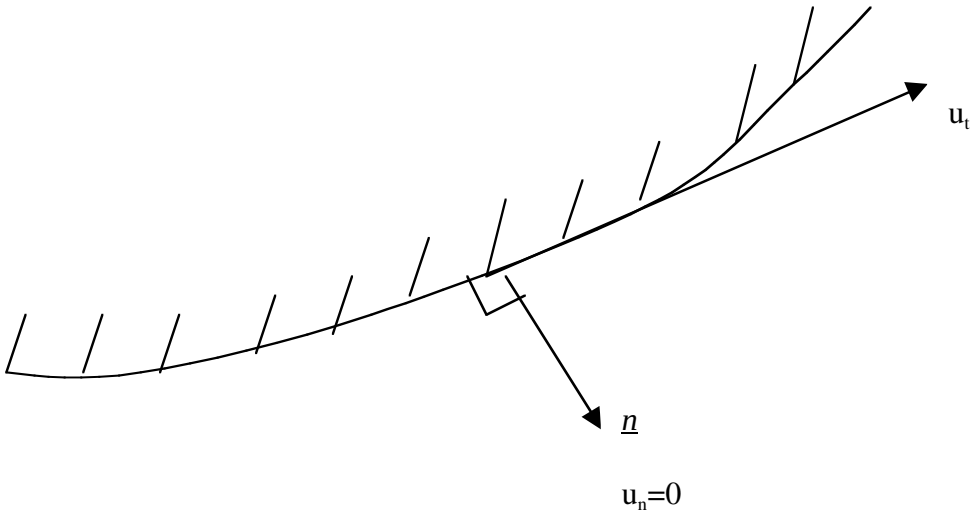
Also note $(\underline{u}.\nabla)\underline{u} = \nabla\left(\frac{1}{2}u^2\right) - \underline{u} \times curl\underline{u}$.

[Proof: $\nabla(\underline{a}.\underline{b}) = (\underline{a}.\nabla)\underline{b} + (\underline{b}.\nabla)\underline{a} + \underline{a} \times curl\underline{b} + b \times curl\underline{a}$].

$$\therefore \frac{\partial \underline{u}}{\partial t} - \underline{u} \times curl\underline{u} = -\frac{1}{\rho}\nabla p - \nabla\left(\frac{1}{2}u^2\right) + \underline{F} \quad \rightarrow \text{ Euler Equation.}$$

Boundary Conditions: ususally specify $\underline{u}$ at enclosing surface → unique solutions.

Fixed Rigid Surface – In real fluid, $\underline{u}=0$ at surface because of friction. Too restrictive for inviscid fluids. Must allow tangential motion. So all we ask is that $\underline{u}.\underline{n}=0$ where $\underline{n}$ is normal to fixed rigid surface.



Collapse of Spherical Vapour Bubble.

Initial State: fluid at rest at pressure $p_\infty (> p_v)$ containing single spherical bubble at pressure $p_V$

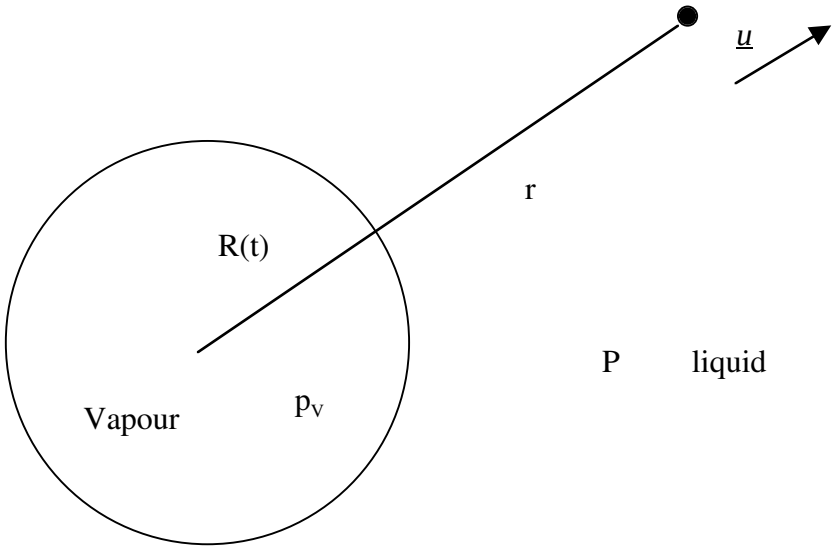
Assume:

(1) Motion remains spherically symmetric – centre of bubble at rest.

(2) Ignore gravity.

(3) Pressure in vapour in bubble always equals $p_V$.

(4) Liquid incompressible.




$\underline{u} = u_r \hat{\underline{r}}$

Continuity Equation. *div* $\underline{u}$=0 ie $\frac{\partial}{\partial r}\left(r^2 u_r\right) = 0$. (r coord of divergence in spherical polars.)

$$[\, div\underline{a} = \frac{1}{r^2 \sin\theta}\left\{\frac{\partial}{\partial r}\left(a_r r^2 \sin\theta\right) + \frac{\partial}{\partial \theta}\left(a_\vartheta r \sin\theta\right) + \frac{\partial}{\partial \phi}\left(a_\phi r\right)\right\}].$$

Euler Equation. $\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} = -\frac{1}{\rho_0}\frac{\partial p}{\partial r}$

From Continuity Equation, $r^2 u_r = F(t)$ only – not a function of r.

Suppose bubble has radius, R(t), then fluid particle on bubble surfacehas velocity $\dot{R}$.

i.e. $F(t) = R^2\dot{R}$ (ignoring mass transfer across surface – fluid particle on surface remains on surface)

$$\therefore u_r(r,t) = \frac{R^2\dot{R}}{r^2}$$ ( for arbitrary fluid particle.)

Substitute in Euler:

$$\frac{\partial\left(\frac{R^2\dot{R}}{r^2}\right)}{\partial t} + \frac{R^2\dot{R}}{r^2}\left(-\frac{2}{r^3}\right)R^2\dot{R} = -\frac{1}{\rho_0}\frac{\partial p}{\partial r}$$

Integrating wrt r

$$-\frac{1}{r}\frac{\partial R^2\dot{R}}{\partial t} + \frac{1}{2}\frac{R^4\dot{R}^2}{r^4} = -\frac{p}{\rho_0} + \chi(t)$$

Let $r \to \infty$. We know $p \to p_\infty$

$$\chi(t) = \frac{p_\infty}{\rho_0}$$

$$\therefore \frac{p_\infty - p}{\rho_0} = \frac{1}{2}\frac{R^4\dot{R}^2}{r^4} - \frac{1}{r}\left(2R\dot{R}^2 + R^2\ddot{R}\right)$$

Put $p=p_V$ at $r=R$ (ignore surface tension)

$$\therefore \frac{p_\infty - p}{\rho_0} = \frac{1}{2}\dot{R}^2 - 2\dot{R}^2 - R\ddot{R}$$

$$\therefore R\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{p_V - p_\infty}{\rho_0} = -p$$

Exercise: if $R=R_0$ at $t=t_0$ and fluid begins at rest, solve above equation to obtain collapse time $t(R) = \left(\frac{3}{2p}\right)^{\frac{1}{2}}\int_R^{R_o}\left(\frac{R^3}{R_o'^3} - 1\right)^{-\frac{1}{2}} dR'$. (Use $\frac{d}{dR}\left(\frac{1}{2}\dot{R}^2\right) = \ddot{R}$)

(If $R_0$=1mm, $\rho_0=10^3$kg m$^{-2}$, $p_\infty = 10^5 Nm^{-2}$, $p_V=0 \Rightarrow$ $t_0=10^{-4}$s.)

2. Bernoulli's Equation.

Start from Euler's Equation:

$$\therefore \frac{\partial \underline{u}}{\partial t} - \underline{u} \times curl\underline{u} + \nabla\left(\frac{1}{2}u^2\right) = -\frac{1}{\rho}\nabla p + \underline{F}.$$

For a conservative, non-dissipative system, neglecting viscosity $\underline{F} = -\nabla V$ where V is the potential energy per unit mass.

$$\therefore \frac{\partial \underline{u}}{\partial t} - \underline{u} \times curl\underline{u} + \nabla\left(\frac{1}{2}u^2\right) = -\frac{1}{\rho}\nabla p - \nabla V$$

Also assume irrotational flow i.e. $curl\underline{u} = 0$ everywhere (see later).

$\Rightarrow$ there exists velocity potential, $\phi$, such that $\underline{u} = \nabla\phi$.

$$\therefore \nabla\left(\frac{\partial \phi}{\partial t}\right) + \nabla\left(\frac{1}{2}u^2\right) + \frac{1}{\rho}\nabla p + \nabla V = 0$$

Form the scalar product with a line element, $\underline{dl}$

[Recall $\nabla\psi.\underline{dl} = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy + \frac{\partial\psi}{\partial z}dz = d\psi$ $\quad\therefore \int_A^B \nabla\psi.\underline{dl} = \int_A^B d\psi = \psi_B - \psi_A.$]

Hence $d\left(\frac{\partial \phi}{\partial t}\right) + d\left(\frac{1}{2}u^2\right) + \frac{dp}{\rho} + dV = 0$

i.e. no change in the above quantity along $\underline{dl}$

$$\int d\left(\frac{\partial \phi}{\partial t}\right) + \int d\left(\frac{1}{2}u^2\right) + \int \frac{dp}{\rho} + \int dV = F(t)$$

(function of time only since spatial differential)

$$\therefore \frac{\partial \phi}{\partial t} + \frac{1}{2}u^2 + \int^p \frac{dp}{\rho} + V = F(t)$$ - Bernoulli's Equation for Irrotational Flow

Thus, in irrotational flow, the Bernoulli sums at any two points are constant at time, t.

Steady, Irrotational Flow, Bernoulli's Equn. $\Rightarrow$ $V + \frac{1}{2}u^2 + \int^p \frac{dp}{\rho} =$constant.

Another form of Bernoulli's Equation holds with *curl* $\underline{u} \neq 0$ if $\underline{dl}$ can be chosen so that $curl\underline{u} \times \underline{u}.\underline{dl} = 0$. In particular in steady flow if dl is taken along a streamline i.e. $\underline{dl}$ is parallel to $\underline{u}$:-

$V + \frac{1}{2}u^2 + \int^p \frac{dp}{\rho} =$constant along streamline.

Particular Cases for $\int \frac{dp}{\rho}$

1. Incompressible, homogeneous fluid. ρ=constant, $\int \frac{dp}{\rho} = \frac{p}{\rho}$

2. Adiabatic, compressible flow. $p = k\rho^\gamma$

$dp = k\gamma\rho^{\gamma-1}d\rho$

$\therefore \int \frac{dp}{\rho} = \int k\gamma\rho^{\gamma-2}d\rho = \frac{k\gamma}{\gamma-1}\rho^{\gamma-1} = \frac{\gamma}{\gamma-1}\frac{p}{\rho} = \frac{1}{\gamma-1}\frac{dp}{d\rho}$

Put $c^2 = \frac{dp}{d\rho}$. When we consider sound waves, we shall identify c with speed of sound.

$\therefore \int \frac{dp}{\rho} = \frac{c^2}{\gamma-1}$

Hence for steady adiabatic flow

$\frac{\gamma}{\gamma-1}\frac{p}{\rho} + V + \frac{1}{2}u^2 =$constant along streamline.

If V=constant, $\frac{\gamma}{\gamma-1}\frac{p}{\rho} + \frac{1}{2}u^2 =$constant along streamline.

Alternative form of Bernoulli.

For ideal gas, $p = \rho \frac{R}{m} T$, [$m$ is MW]

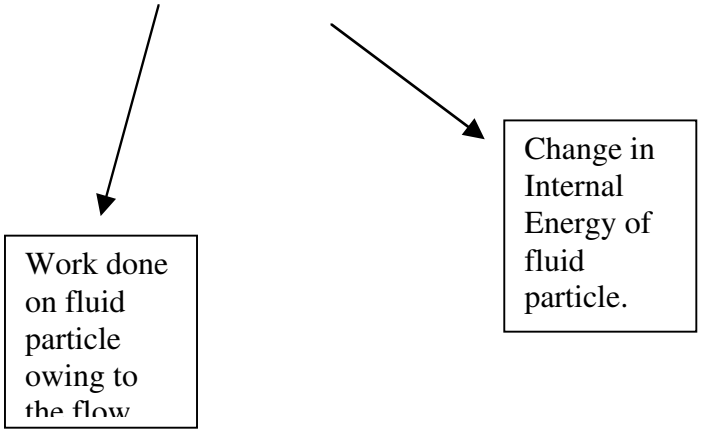
and $E = c_V T = \frac{c_V}{\frac{R}{m}} \frac{p}{\rho}$ [$dQ = dE + pdV$ ( $= c_V dT$)]

$= \frac{c_V}{c_P - c_V} \frac{p}{\rho}$ [since $c_P - c_V = \frac{R}{m}$]

$= \frac{1}{\gamma - 1} \frac{p}{\rho}$ [$\gamma = \frac{c_P}{c_V}$]

where E is the internal energy of the gas, $c_V$, $c_P$ are specific heats at constant V, p.

Now $\frac{\gamma}{\gamma - 1} \frac{p}{\rho} = \frac{p}{\rho} + \frac{1}{\gamma - 1} \frac{p}{\rho} = \frac{R}{m} T + c_V T = c_P T$




$\therefore \frac{1}{2} u^2 + V + c_P T =$ constant (along streamline)

Alternative form of Bernoulli Equation for steady adiabatic flow.