## 8. Subsonic and Supersonic Motion.

Considering high speed, compressible flows. Assume steady irrotational, inviscid, adiabatic/reversible (i.e. homentropic) flow.

Bernoulli (steady flow) $\rightarrow \quad \frac{1}{2}u^2 + \int \frac{dp}{\rho} + V = \text{ constant}$ (everywhere).

$\Rightarrow \qquad \frac{1}{2}u^2 + \frac{c^2}{\gamma - 1} + V = \text{constant}$ (adiabatic)

Assume V=constant (body force=0) $\quad \therefore \frac{1}{2}u^2 + \frac{c^2}{\gamma - 1} = \text{constant}$

$$\therefore \frac{1}{2}u^2 = \frac{1}{\gamma - 1}\left(c_0^2 - c^2\right)$$

where $c=c_0$ when u=0. We call $c_0$ the stagnation speed of sound. u increases as c decreases from $c_0$.

The maximum value of u occurs when c=0. Hence $u_{\max} = c_0\left(\frac{2}{\gamma - 1}\right)^{\frac{1}{2}} \quad (\gamma > 1)$.

$u_{\max}$ is called the LIMIT or CAVITATION SPEED. $c = 0 \Rightarrow c^2 = \frac{dp}{d\rho} = k\gamma\rho^{\gamma - 1} = 0$

i.e. $\rho \rightarrow 0$, $p \rightarrow 0 \Rightarrow u_{\max}$ obtained if gas flowing out to a vacuum. Note c, ρ, p decrease as u increases.

The flow is called SONIC at some point if u=c there. We denote this speed by $c^*$ and call it the SONIC SPEED.

Here $\frac{1}{2}c^{*2} = \frac{1}{\gamma - 1}\left(c_0^2 - c^{*2}\right)$. $\quad \therefore c^* = c_0\left(\frac{2}{\gamma + 1}\right)^{\frac{1}{2}}$.

Hence $u_{\max} = c^*\left(\frac{\gamma + 1}{\gamma - 1}\right)^{\frac{1}{2}}$.

We derive an alternative form of Bernoulli's Equation required for shocks.

For adiabatic flow, $p=k\rho^{\gamma}$, $c^2 = \frac{dp}{d\rho} = \frac{\gamma p}{\rho}$.

$$c_0^2 = \frac{1}{2}(\gamma+1)c^{*2}$$

$$\therefore \frac{1}{2}u^2 + \frac{c^2}{\gamma-1} = \frac{c_0^2}{\gamma-1} \quad \text{becomes} \quad \underline{\frac{1}{2}u^2 + \frac{\gamma}{\gamma-1}\frac{p}{\rho} = \frac{1}{2}\left(\frac{\gamma+1}{\gamma-1}\right)c^{*2}}.$$

Also from Bernoulli's Equation.:-

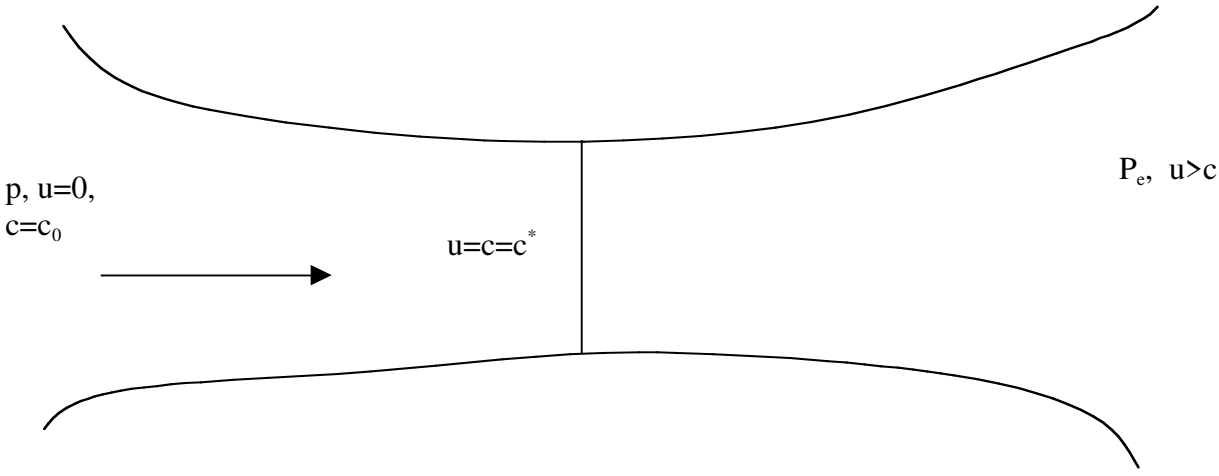
$$c^2 = c_0^2 - \frac{1}{2}(\gamma-1)u^2$$

$$\therefore c^2 - u^2 = c_0^2 - \frac{1}{2}(\gamma+1)u^2 = \frac{1}{2}(\gamma+1)\left(c^{*2} - u^2\right).$$

Hence $u>c^*$ or $u<c^*$ according to whether $u>c$ or $u<c$.

($u>c^*$ can be more convenient - don't need to work out c locally.)

<u>Simplest Example of Achieving Sonic Speed – Tube Flow.</u>

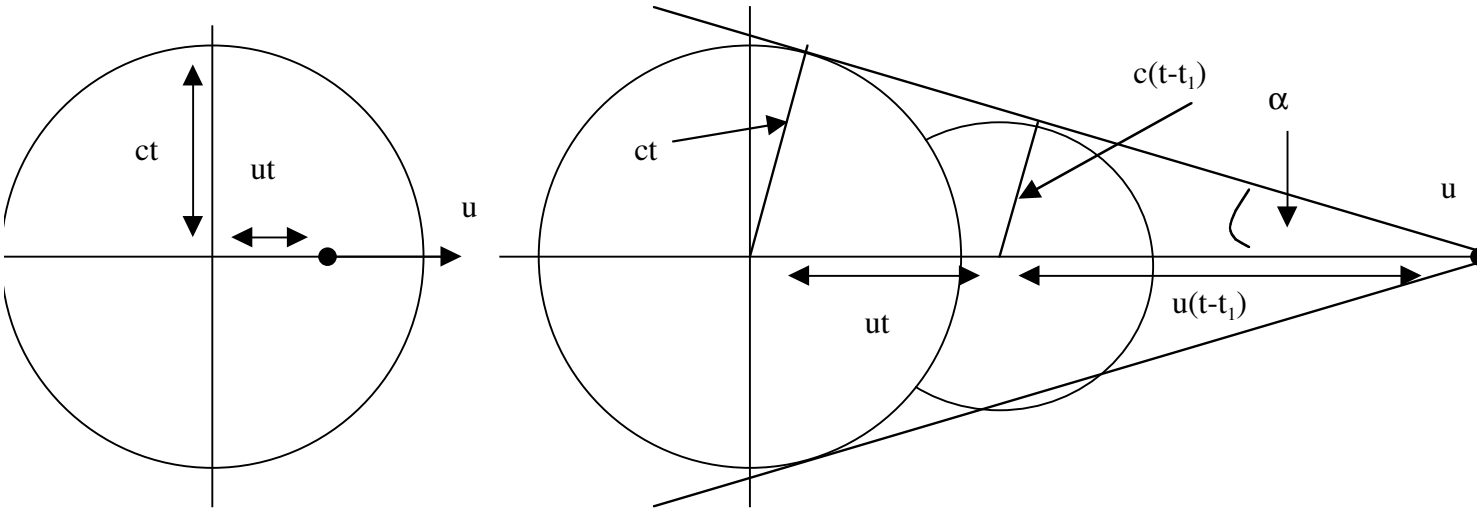



For $p>p_e$, u=c at throat u>c towards exit and shock can develop (high u $\Rightarrow$ low p$\rightarrow$ misses $p_e$.)

2-D Subsonic and Supersonic Flow.

Convenient to measure speed of flow by MACH NUMBER, $M = \frac{u}{c}$.

Subsonic, M<1. Supersonic M>1.

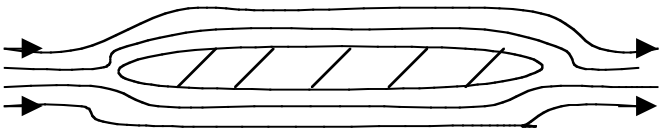



| (a) Source is always inside circular sound wave front. ∴ always hear source sound before source arrives- Sound wave propagates to all parts of flow. | (b) Source is always outside circular sound wave front. ∴ don't hear source before it arrives. If source fixed and air moves with speed -u then no propagation of sound upstream of source. |
|---|---|

α is MACH ANGLE, defining MACH CONE. $\sin\alpha = \frac{ct}{ut} = \frac{1}{M}$.

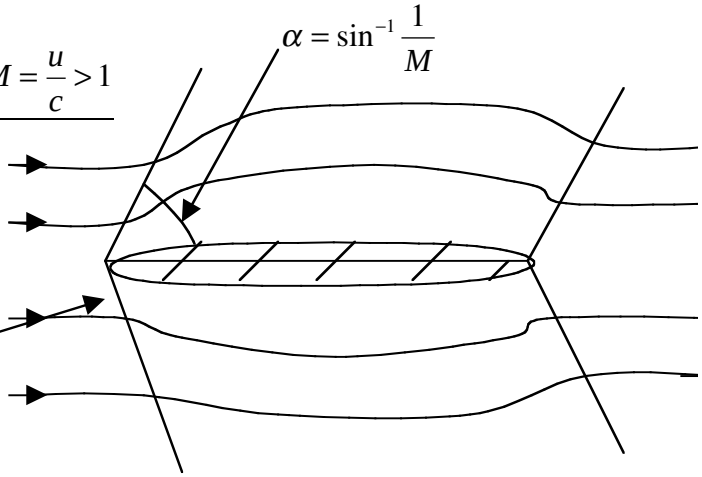
2-D Subsonic and Supersonic Flow Past Thin Aerofoil.

$M = \frac{u}{c} < 1$

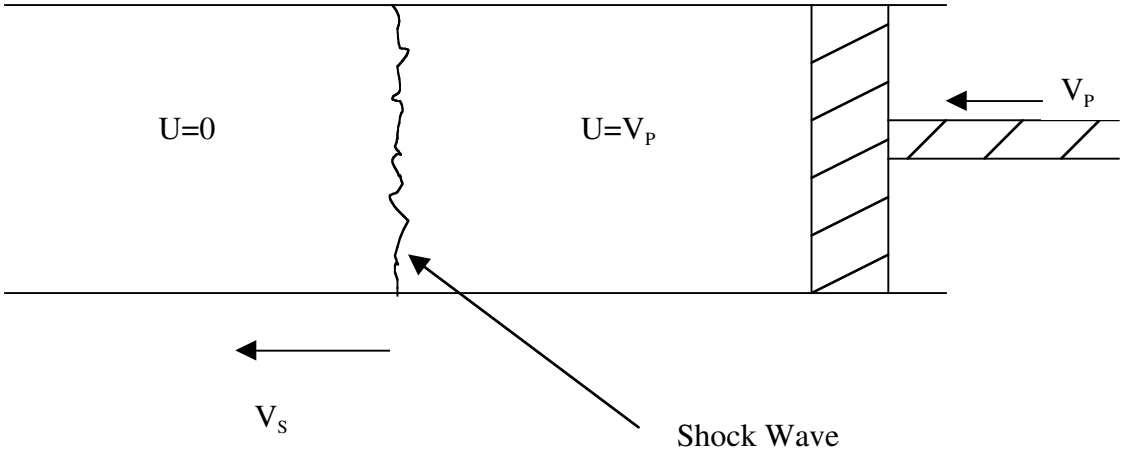


Flow unaffected outside MACH LINES in Supersonic Flow.

$M = \frac{u}{c} > 1$

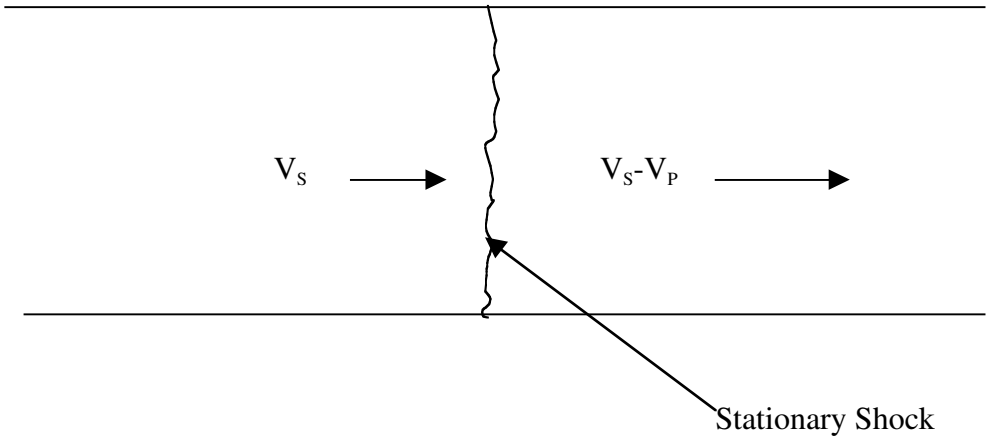


9. 1-Dimensional Shock Wave.

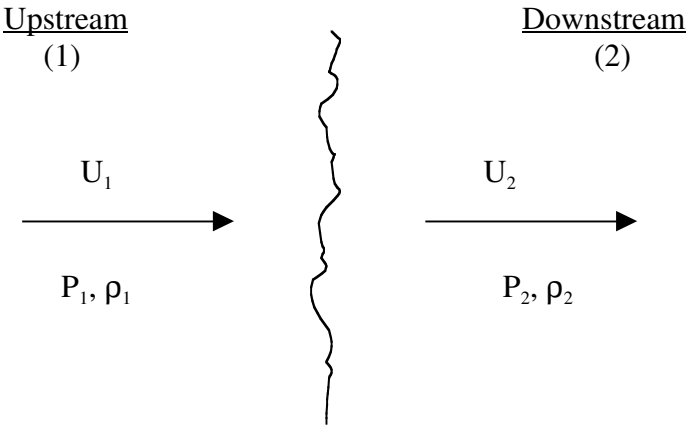



Piston moving in gas cylinder at high speed, $V_P$, forms shock wave. Shock discontinuities occur in distances of $10^{-4}$cm so can model shock thickness as being negligible. Model consists of plane normal to flow with conditions uniform but different on either side. Assume adiabatic conditions nand non-viscous fluid. Model is idealisation – technically shock can't form if inviscid.

No loss of generality by considering stationary, 1-D shock.



1-D Stationary Shock.




Consider unit area of shock front. In time, δt, a mass $\rho_1 u_1 \delta t$ crosses through with momentum equal to $\rho_1 u_1^2 \delta t$. On the other side, this becomes a mass, $\rho_2 u_2 \delta t$, with momentum, $\rho_2 u_2^2 \delta t$. Since mass is conserved:-

$$\rho_1 u_1 = \rho_2 u_2 \qquad (1)$$

Difference in pressure on either side must equal rate of change of momentum:-

$$\therefore p_1 - p_2 = \rho_2 u_2^2 - \rho_1 u_1^2 \qquad (2) \qquad \left[ F = \frac{\delta(mv)}{\delta t} \right]$$

$$\therefore p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2 \qquad (3)$$

Combine (1) and (2) :-

$$p_1 - p_2 = \rho_1 u_1 u_2 - \rho_2 u_2 u_1 = (\rho_1 - \rho_2) u_1 u_2$$

$$\therefore u_1 u_2 = \frac{p_1 - p_2}{\rho_1 - \rho_2} \qquad (4)$$

Flow is adiabatic. Bernoulli's Equation in absence of external forces for regions (1) and (2) are :-

$$\frac{1}{2}u_1^2 + \frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1} = \left(\frac{\gamma+1}{\gamma-1}\right)\frac{1}{2}c_1^{*2}$$

$$\frac{1}{2}u_2^2 + \frac{\gamma}{\gamma-1}\frac{p_2}{\rho_2} = \left(\frac{\gamma+1}{\gamma-1}\right)\frac{1}{2}c_2^{*2}$$

Conservation of Energy implies that LHS of these equations must be equal, hence $c_1^* = c_2^* = c^*$.

$$\therefore \frac{1}{2}u_1^2 + \frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1} = \frac{1}{2}u_2^2 + \frac{\gamma}{\gamma-1}\frac{p_2}{\rho_2} = \left(\frac{\gamma+1}{\gamma-1}\right)\frac{1}{2}c^{*2} \qquad (5)$$

Rewrite equation (5):

$$(\gamma+1)c^{*2}\rho_1 = (\gamma-1)\left(\rho_1 u_1^2 + p_1\right) + (\gamma+1)p_1$$
$$(\gamma+1)c^{*2}\rho_2 = (\gamma-1)\left(\rho_2 u_2^2 + p_2\right) + (\gamma+1)p_2$$

Subtract and use equation (3):- $(\gamma+1)c^{*2}(\rho_1 - \rho_2) = (\gamma+1)(p_1 - p_2)$

$$c^{*2} = \frac{p_1 - p_2}{\rho_1 - \rho_2} = u_1 u_2 \qquad (6)$$

$\underline{u_1 u_2 = c^{*2}}$ is known as PRANDTL'S RELATION.

$\Rightarrow$ if $\underline{u_1 > c^*}$ then $\underline{u_2 < c^*}$ (or vice versa – ruled out later!)

Now derive RANKINE-HUGONIOT Relation for p, ρ across shock.

From (5) $u_2^2 - u_1^2 = \frac{2\gamma}{\gamma-1}\left(\frac{p_1}{\rho_1} - \frac{p_2}{\rho_2}\right)$.

From (1) and (4):- $\frac{u_1^2}{\rho_2^2} = \frac{u_2^2}{\rho_1^2} = \frac{u_1 u_2}{\rho_2 \rho_1}$ from (1)

$= \frac{1}{\rho_2 \rho_1}\left(\frac{p_1 - p_2}{\rho_1 - \rho_2}\right)$ from (4).

$\therefore \quad u_1^2 = \frac{\rho_2}{\rho_1}\left(\frac{p_1 - p_2}{\rho_1 - \rho_2}\right)$ , $u_2^2 = \frac{\rho_1}{\rho_2}\left(\frac{p_1 - p_2}{\rho_1 - \rho_2}\right)$.

$\Rightarrow \quad u_2^2 - u_1^2 = \left(\frac{p_1 - p_2}{\rho_1 - \rho_2}\right)\left(\frac{\rho_1}{\rho_2} - \frac{\rho_2}{\rho_1}\right) = (p_1 - p_2)\left(\frac{\rho_1 + \rho_2}{\rho_1 \rho_2}\right) = \frac{2\gamma}{\gamma-1}\left(\frac{p_1}{\rho_1} - \frac{p_2}{\rho_2}\right)$ from (5).

$\therefore$ Re-writing to get compression ratio $\frac{\rho_2}{\rho_1}$ in terms of pressure ratios:-

$$\underline{\frac{\rho_2}{\rho_1} = \frac{(\gamma-1)p_1 + (\gamma+1)p_2}{(\gamma+1)p_1 + (\gamma-1)p_2} \quad \left(= \frac{u_1}{u_2}\right)} \quad (7) \quad \text{(RANKINE-HUGONIOT RELATION)}$$

(7) in general inconsistent with adiabatic law $\frac{\rho_2}{\rho_1} = \left(\frac{p_2}{p_1}\right)^{\frac{1}{\gamma}}$.

Hence flow through shock front is NOT isentropic (adiabatic $\Rightarrow$ dQ=0 but irreversible $\Rightarrow \quad dS > \frac{dQ}{dT} = 0$).

To find which region supersonic:

(5) can be written:-

$$\frac{1}{2}u_1^2 + c_p T_1 = \frac{1}{2}u_2^2 + c_p T_2 \qquad (8) \quad \text{(derived previously).}$$

So temperature where $u<c^*$ must be greater than temperature where $u>c^*$.

$T_2>T_1$ follows from 2nd Law of Thermodynamics – general result but only proven here in the case of a weak shock.

Weak Shock.

Derive entropy change across weak shock. Let $\frac{p_2}{p_1} = 1 + \Delta$.

From (7) $\frac{\rho_2}{\rho_1} = \frac{(\gamma-1)+(\gamma+1)(1+\Delta)}{(\gamma+1)+(\gamma-1)(1+\Delta)} = \frac{1+\left(\frac{\gamma+1}{2\gamma}\right)\Delta}{1+\left(\frac{\gamma-1}{2\gamma}\right)\Delta}$.

Entropy difference in terms of pressures and densities is :-

$$\frac{p_2}{p_1}\left(\frac{\rho_1}{\rho_2}\right)^{\gamma} = \exp\left(\frac{S_2 - S_1}{c_V}\right)$$

$$\therefore \quad \frac{S_2 - S_1}{c_V} = \ln\frac{p_2}{p_1} + \gamma \ln\frac{\rho_1}{\rho_2} = \ln(1+\Delta) + \gamma\ln\left(1+\frac{\gamma-1}{2\gamma}\Delta\right) - \gamma\ln\left(1+\frac{\gamma+1}{2\gamma}\Delta\right)$$

$$= \left(\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} + \ldots\right) + \gamma\left[\left(\frac{\gamma-1}{2\gamma}\right)\Delta - \left(\frac{\gamma-1}{2\gamma}\right)^2\frac{\Delta^2}{2} + \left(\frac{\gamma-1}{2\gamma}\right)^3\frac{\Delta^3}{3} + \ldots\right]$$

$$-\gamma\left[\left(\frac{\gamma+1}{2\gamma}\right)\Delta - \left(\frac{\gamma+1}{2\gamma}\right)^2\frac{\Delta^2}{2} + \left(\frac{\gamma+1}{2\gamma}\right)^3\frac{\Delta^3}{3} + \ldots\right] \qquad \text{(assuming } \Delta << 1\text{)}$$

$$= \underline{\frac{\gamma^2-1}{12\gamma^2}\Delta^3 + \ldots}$$

Since $\gamma > 1$ and $S_2 > S_1$ by the Second Law $\Rightarrow \quad \Delta > 1 \quad \Rightarrow \quad \underline{p_2 > p_1}$.

Now from (6): $\frac{p_1 - p_2}{\rho_1 - \rho_2} = c^{*2}$. $\therefore$ if $p_2 > p_1$, then $\rho_2 > \rho_1$.

From (1), $\rho_2 > \rho_1 \Rightarrow u_1 > u_2$ ie upstream region is SUPERSONIC and downstream region is SUBSONIC. From (8), $T_2 > T_1$.

Summary: $\rho_2 > \rho_1$, $p_2 > p_1$, $S_2 > S_1$, $T_2 > T_1$, $u_2 < u_1$. $u_1$ is supersonic, $u_2$ is subsonic.
All shock waves are compression shocks. Heat is generated across a shock. Viscous forces convert KE of ordered flow into thermal energy, causing entropy to rise across a shock.

N.B. From (7) $\frac{\rho_2}{\rho_1} \xrightarrow[\frac{p_2}{p_1}\to\infty]{} \frac{\gamma+1}{\gamma-1}$ eg for diatomic gas, $\gamma = \frac{7}{5}$, $\frac{\gamma+1}{\gamma-1} = \frac{12/5}{2/5} = 6$ ie compression ratio across shock remains finite.