Level 4 Hydrodynamics

Course Structure

1. Basic Equaitions
2. Bernoulli Equation
3. Concept of Vorticity
4. Hydrostatics
5. Inviscid and Incompressible Flow
6. Compressible Fluids
7. Entropy of a Fluid
8. Subsonic and Supersonic Motion
9. 1-D Shock Wave
10. Gravitational Instability and Galaxy Formation

Books

"A First Course in Fluid Dynamics", A.R. Paterson, CUP

"Elementary Fluid Dynamics", D.J. Acheson, Oxford App. Maths

"An Introduction to Fluid Dynamics", G.K. Batchelor, CUP.

Hydrodynamics - the study of the behaviour of fluids (liquids and gases). We shall assume the fluid to be continuous(mean free path of a molecule << linear dimension of system being studied.) Describe state of fluid by vector and scalar fields.

Applications

1. Aerodynamics

2. Studies of waves in continuous media (eg atmosphere, Earth's crust, interstellar space).

3. Galaxy formation and clustering.

Quite mathematical; main requirements are vector calculus, complex variables and theory of partial differential equations.

In first instance, neglect viscosity. Viscosity is essentially friction – easier to move knife through treacle than water, even edge-on. Viscous force acts tangentially to surface.

1. Basic Equations

Description of Motion.

Usually describe fluid flow by means of expression $\underline{u} = \underline{u}(\underline{X}, t)$ for the flow velocity at any point $\underline{X}$ and at any time t. Finding $\underline{u}(\underline{X}, t)$ is usually the main aim in the "Eulerian" approach.

1. Euler uses $\underline{u}(\underline{X}, t)$ where $\underline{u}$ is velocity of fluid particle at $\underline{X}$ at t.

$$\underline{u}(\underline{X}, t) \equiv [u(x, y, z, t), v(x, y, z, t), w(x, y, z, t)]$$

("Flowmeters, fixed in riverbed at each $\underline{X}$, give flow velocity for all t")

2. Lagrange. Suppose fluid is at $\underline{a}$ at time $t=t_o$ and suppose it is at $\underline{X}$ at time t $\rightarrow$ $\underline{X}(\underline{a},t)$ will specify motion, (as will $\frac{d\underline{X}(\underline{a},t)}{dt} = \underline{u}(\underline{a},t)$).

("Following motion of leaf in stream")

Adopt Eulerian approach from now on.

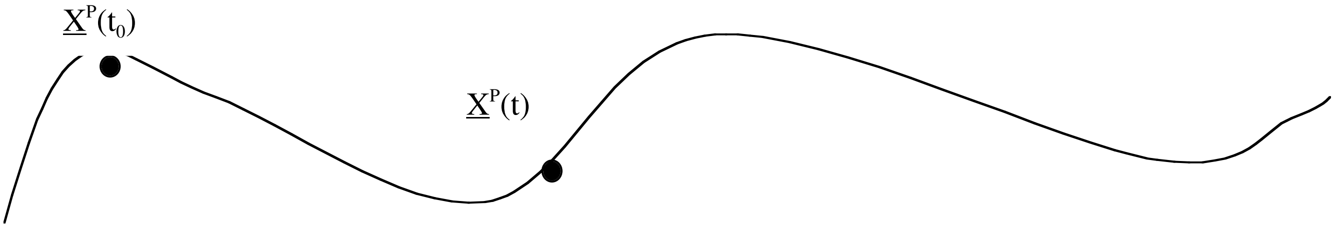
Velocity Field $\underline{u}(\underline{X},t)$ If velocity field, $\underline{u}$, is independent of t – steady motion. If w=0 and u, v independent of z, motion is called plane or 2-D. Think of block between z=0 and z=1 filled with fluid.

Density is $\rho(\underline{x},t)$. $\underline{u}$, ρ assumed differentiable in $\underline{X}$ and t.

Visualising flow solutions: Streamlines and Particle paths – different in general..

1) Particle paths. Trajectory of fluid particles through fluid.

Suppose fluid particle has coords $\underline{X}^P(t)$



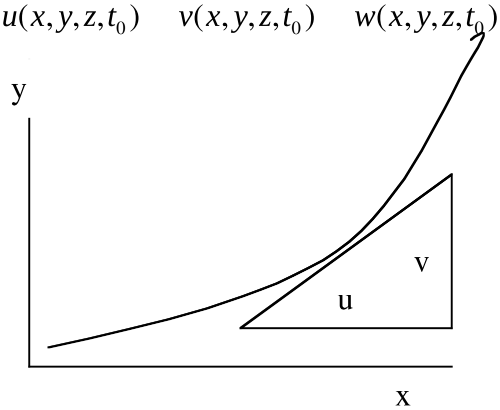
At any instant, fluid particle has velocity of fluid at $\underline{X}^P$

$$\frac{d\underline{X}^P}{dt} = \underline{u}(\underline{X}^P,t)$$

Finding particle paths involves integrating ordinary differential equations.

2) Instantaneous streamlines. Curves whose paths are parallel to velocity $\underline{u}$ at (x,y,z) at time t0.

$$\frac{dx}{u(x,y,z,t_0)} = \frac{dy}{v(x,y,z,t_0)} = \frac{dz}{w(x,y,z,t_0)}$$



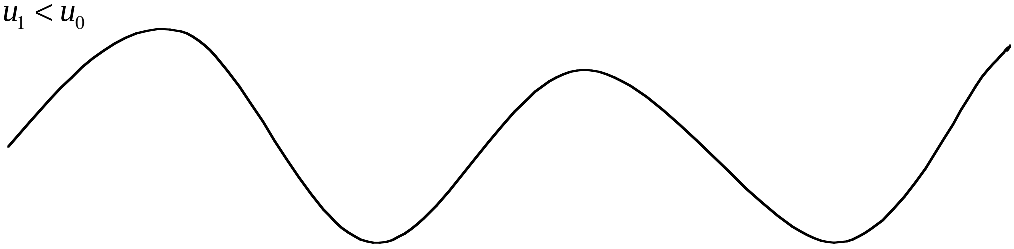
→ Instantaneous "snapshot" of direction of motion of each particle at t_0. (river flow VF).

Example. 2-D Flow: $\underline{u} = (u_0 + u_1 \cos(t), u_1 \sin(t))$

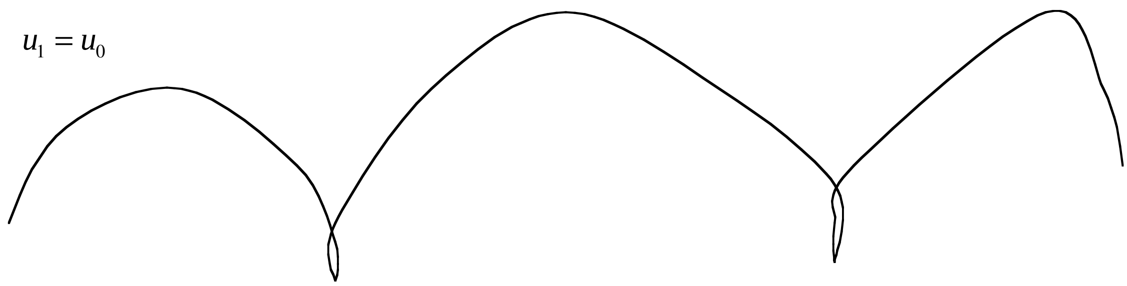
Particle Paths. $\dfrac{dX^P}{dt} = u_0 + u_1 \cos(t), \quad X^P = X^P(0) + u_0 t + u_1 \sin(t)$

$$\frac{dY^P}{dt} = u_1 \sin(t), \qquad Y^P = Y^P(0) - u_1 \cos(t)$$

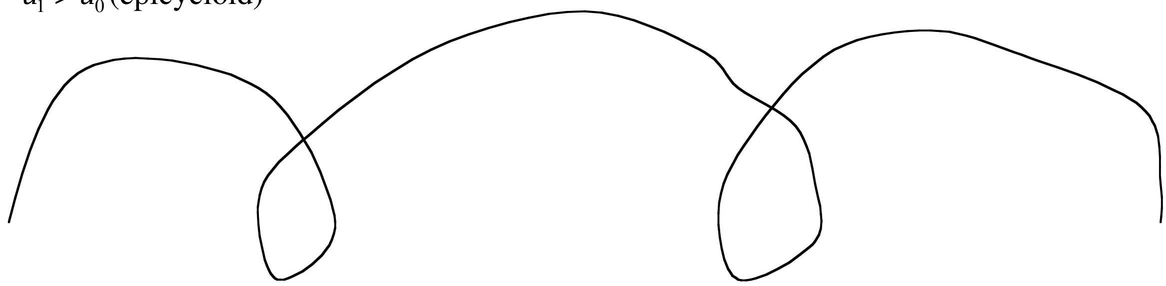
$u_1 < u_0$



$u_1 = u_0$

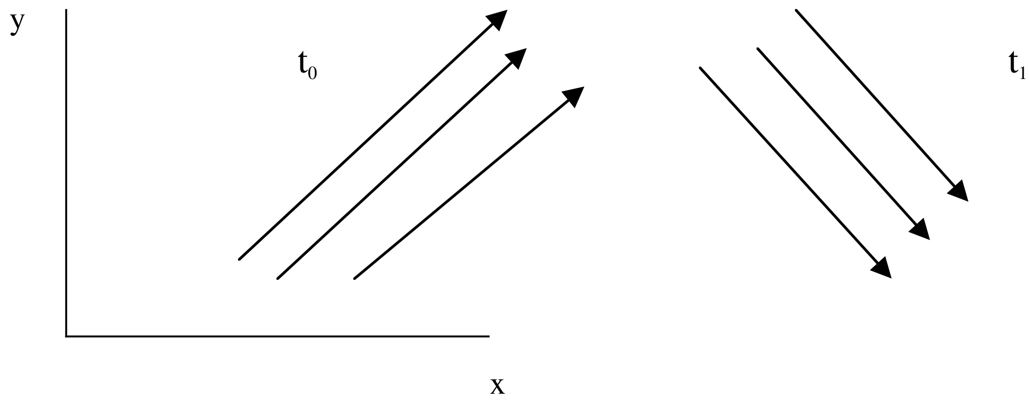


$u_1 > u_0$ (epicycloid)



Streamline
$$\frac{dx}{u_0 + u_1 \cos(t_0)} = \frac{dy}{u_1 \sin(t_0)}$$

$$\therefore y = \frac{u_1 \sin(t_0)}{u_0 + u_1 \cos(t_0)} x + c$$



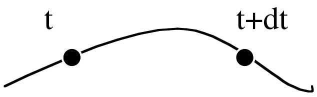
Inatantaneous snapshots of fluid flow →"streamlines"

NB streamlines and particle paths different because not steady flow.

Exercise For velocity field $\underline{u} = (ay, -a(x - bt), 0)$ find (a) the particle paths (b) the streamlines. (see Chapter 3 of Paterson).

Material Derivative

Suppose flow contains a temperature field, $\Theta(\underline{X},t)$. Suppose fluid particle is at (x,y, z) at



time t where it experiences a temperature $\Theta(x,y,z,t)$. At t+dt it experiences $\Theta(x+dx,y+dy,z+dz,t+dt)$

$$\Theta(x,y,z,t)+\frac{\partial\Theta}{\partial x}dx+\frac{\partial\Theta}{\partial y}dy+\frac{\partial\Theta}{\partial z}dz+\frac{\partial\Theta}{\partial t}dt.$$

$$\therefore\ \text{Change in }\Theta=\frac{\partial\Theta}{\partial x}dx+\frac{\partial\Theta}{\partial y}dy+\frac{\partial\Theta}{\partial z}dz+\frac{\partial\Theta}{\partial t}dt.$$

Now,

$$dx=u(x,y,z,t)dt$$

$$dy=v(x,y,z,t)dt$$

$$dz=w(x,y,z,t)dt$$

$$\therefore\ \text{Change in}\quad \Theta=dt\left(\frac{\partial\Theta}{\partial t}+u\frac{\partial\Theta}{\partial x}+v\frac{\partial\Theta}{\partial y}+w\frac{\partial\Theta}{\partial z}\right)$$

$\therefore$ Rate of change of Θ at fluid particle

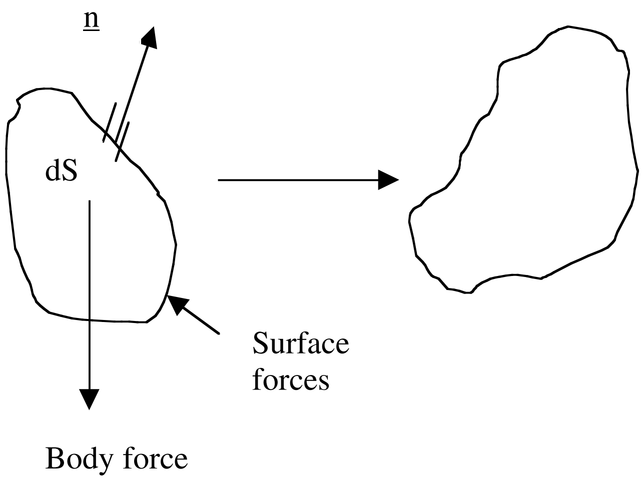
$$=\frac{\partial\Theta}{\partial t}+u\frac{\partial\Theta}{\partial x}+v\frac{\partial\Theta}{\partial y}+w\frac{\partial\Theta}{\partial z}$$
$$=\frac{\partial\Theta}{\partial t}+(\underline{u}.\nabla)\Theta$$
$$=\frac{D\Theta}{Dt}$$

$\rightarrow$ Material or Convective Derivative

Get change even if $\frac{\partial\Theta}{\partial t}=0$, because of change of position of fluid particle. Can apply to a vector field.

Equation of Motion

Consider forces on a fluid in volume V, bounded by a surface, S (fixed set of fluid particles).



Apply Newton's Second Law:

$$\int_V \rho \underline{f} dV = \int_V \rho \underline{F} dV + \int_S \text{ forces on surface of V}$$

where $\underline{f}(\underline{x},t)$ is acceleration of fluid particle at $\underline{X}$ and t and $\underline{F}(\underline{x},t)$ is body force ie force/unit mass acting on fluid particle from "outside" eg gravity.

Force on $\underline{dS}$ due to fluid outside V = -p $\underline{n}$ dS (Inviscid)

where $\underline{n}$ is outward normal, p is force/unit area (pressure) and $p = p(\underline{X},t)$, independent of $\underline{n}$.

So:

$$\int_V \rho \underline{f} dV = \int_V \rho \underline{F} dV - \int_S p \underline{dS}$$

$$= \int_V \rho \underline{F} dV - \int_V \nabla p dV$$

$$\left[\int \nabla p dV = \int p \underline{dS} \text{ since } \int \nabla p dV = \int \underline{i} \frac{\partial p}{\partial x} dxdydz + ... = \int p \underline{i} dydz + ... = \int p \underline{dS} \right]$$

So $\rho \underline{f} = \rho \underline{F} - \nabla p$

Now $\underline{f} = \frac{D\underline{u}}{Dt}$ (since following fluid particle)

$$= \frac{\partial \underline{u}}{\partial t} + (\underline{u}.\nabla)\underline{u}$$

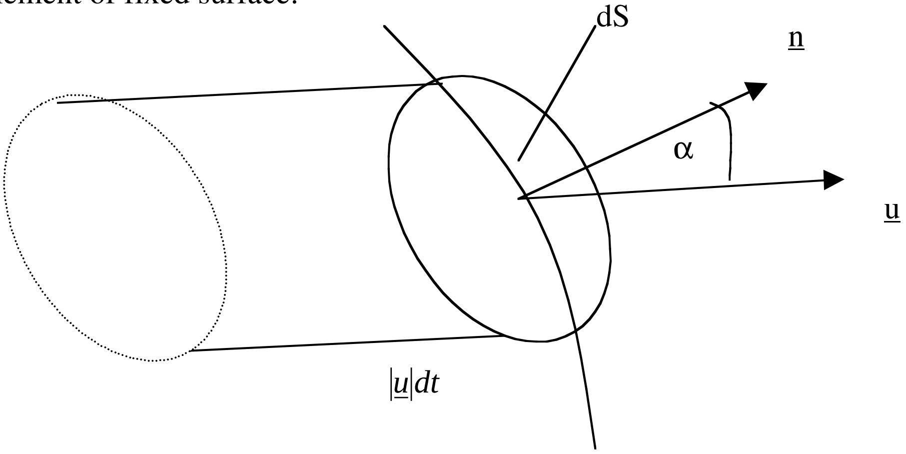
i.e. $\underline{\frac{\partial \underline{u}}{\partial t} + (\underline{u}.\nabla)\underline{u} = \underline{F} - \frac{1}{\rho}\nabla p}$ <u>Euler Equation of Motion for Inviscid Flow</u>.

Even in steady flow where $\underline{u} = \underline{u}(x, y, z)$ and $\frac{\partial \underline{u}}{\partial t} = 0$, the acceleration of an individual fluid particle is non- zero because the velocity field $\underline{u}$ changes as fluid particle moves from one point to another.

Continuity Equation - Conservation of Mass.

Flux of mass across a fixed surface = Mass crossing per unit time

Consider small element of fixed surface:

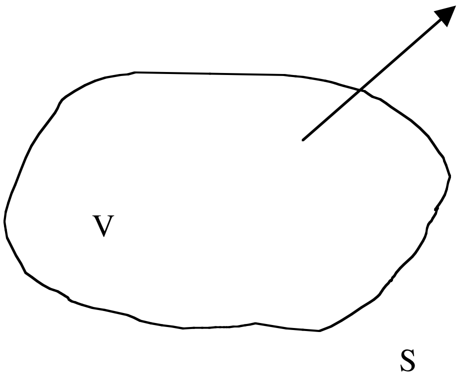


All fluid particles crossing dS in interval (t,t+dt) lie inside cylinder.

$\therefore$ Mass in cylinder $= \rho\left(\underline{X},t\right)\left|\underline{u}\right| dt dS\cos\alpha = \rho\left(\underline{X},t\right) dt\underline{u}.\underline{dS}$

$\therefore$ Mass crossing surface in (t,t+dt) = dt $\int_S \rho\underline{u}.\underline{dS}$

Consider arbitrary closed surface, bounding volume V, fixed in space, n outward normal.



$\therefore$ Mass in V = $\int \rho dV$

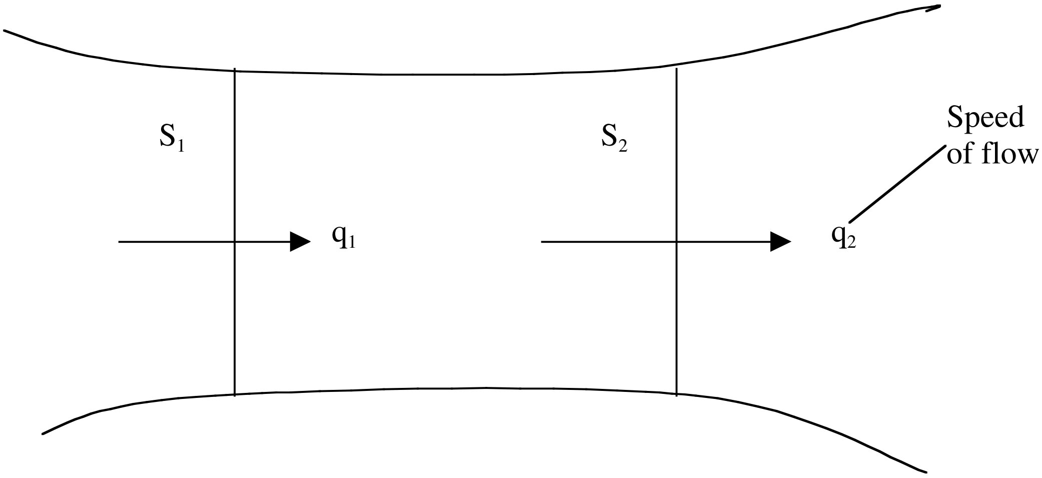
$\therefore$ Mass Conservation $\Rightarrow$ $\underline{\frac{\partial}{\partial t}\int_V \rho dV = -\int_S \rho\underline{u}.\underline{dS}}$

(integral form of continuity equation)

Assume steady motion $\Rightarrow$ $\int \rho\underline{u}.\underline{dS} = 0$.

Also assume incompressible $\Rightarrow$ ρ=constant, $\Rightarrow$ $\int_S \underline{u}.\underline{dS} = 0$.

Example. Incompressible Flow



$\int_S \underline{u}.\underline{dS} = 0 \Rightarrow -q_1 S_1 + q_2 S_2 = 0$

$\therefore S_1 q_1 = S_2 q_2$

ie q increases as 1/S $\Rightarrow$ narrower the gap, faster the flow.

By Divergence Theorem: $\int_S \rho \underline{u}.\underline{dS} = \int_V div(\rho \underline{u}) dV$

$\therefore \int_V \left(\frac{\partial \rho}{\partial t} + div(\rho \underline{u}) \right) dV = 0$

Since V arbitrary and fixed:-

$\underline{\frac{\partial \rho}{\partial t} + div(\rho \underline{u}) = 0}$ $\rightarrow$ Equation of Continuity.

Steady Motion $\Rightarrow$ $\underline{u} = \underline{u}(\underline{X}), \rho = \rho(\underline{X})$ $\therefore$ $\underline{div(\rho \underline{u}) = 0}$.

Incompressible Fluid $\Rightarrow$ $\rho = \rho_0$ $\therefore$ $\underline{div(\underline{u}) = 0}$. (whether steady or not)
(except at source)

Basic hydrodynamic equations:

$$\frac{D\underline{u}}{Dt} = \underline{F} - \frac{1}{\rho}\nabla p \qquad \text{Euler Equation}$$

$$\frac{\partial \rho}{\partial t} + div(\rho \underline{u}) = 0 \qquad \text{Continuity Equation}$$

$\Rightarrow$ 4 equations in 5 unknowns: u,v,w,ρ, p.

Equation of State is 5th equation.

$$p = p(\rho, T)$$

[$= \rho \frac{R}{m} T$ for ideal gas where R is universal gas constant, m is molecular weight of gas.]

T is a further unknown but can be neglected in 3 cases:

1) Infinite Conductivity, $T=T_0$ ie isothermal flow.

$$p = p(\rho, T_0).$$

2) Zero Conductivity + perfect gas ie adiabatic flow

$$p = p(\rho) = k\rho^{\gamma} \qquad \text{(aerodynamics – quick pressure changes)}$$

where $\gamma = \frac{C_p}{C_V}$. (ratio of specific heats at constant p, V)

$\gamma = \frac{7}{5}$ for diatomic gas (air), $\gamma = \frac{5}{3}$ for monatomic gas (plasma).

3) Incompressible Flow

Equation of State: ρ=constant,

(ie density variations produced by p, T variations are small.)